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Folding and Geometry: Buckminster Fuller’s Provocation of Thinking

1. Introduction

From the 19th century to the early 20th century, geometry had changed its character considerably. Discoveries, such as non-Euclidean geometry, alongside the development of differential geometry with its definition of the manifold, instigated a plurality of geometries. Encompassing this plurality was a trend of thought that called for situating geometry on stable foundations, one might say even static pre-determined ones. Against this backdrop of a growing move towards axiomatization, Richard Buckminster “Bucky” Fuller (1895–1983), an American architect, designer and inventor, offered a critic of Euclidean and Cartesian geometry from a novel reconsideration of practical actions and operations like folding – folding, as a form of thinking on and through movement, enabling a different conception of geometry. This paper aims to show that beyond an axiomatized motionless geometry, on the one hand, and the various forgotten mathematizations of the fold, on the other, Fuller suggests to think of movement from a different perspective: movement as the provocation of thinking. It is what provokes and initiates thinking itself. Starting with Fuller’s critique of geometry and concluding with his conception of mobility, we examine notions of movement present in Fuller’s thought. Indeed, folds and folding lie at the core of Fuller’s work as an example of mastering movement.

2. Fuller and Geometry: Fuller's critique and the conception of geometry and folding at the beginning of the 20th century

Before turning to Fuller's conceptions of the fold and mobility, as what provokes stable structures and buildings, we examine Fuller's critique of the axiomatic conception of geometry, as exemplified in Euclidean axiomatics. We then review the manner in which geometry in general and folding in particular were perceived within mathematics, from the beginning of the 19th century till the middle of the 20th century, in order to assess correctly Fuller's critique of the problematic relation between movement and geometry and his conception of folding.

2.1 Fuller and the Euclidean geometry

Needless to say, Euclid's geometry – as presented in his book *Elements* – is one of the most influential theories of western civilization. However, little is known about the author, beyond the fact that he lived in Alexandria around 300 BCE. Most of the theorems appearing in the *Elements* were not discovered by Euclid himself, but were the work of earlier Greek mathematicians such as mathematicians of the Pythagorean School, Hippocrates of Chios, Theaetetus of Athens and Eudoxus of Cnidos. Credited to Euclid is the arrangement of these theorems in a logical manner, in order to show that they necessarily follow from basic definitions, postulates and axioms.¹ The geometrical constructions employed in the *Elements* are restricted to those achieved by using a straightedge and a compass. Empirical proofs using measurement were not allowed: i.e., the only statements that were allowed were these in form of declaring that magnitudes are either equal, or that one is greater than the other.

Euclid's rigor and organization was admired throughout the ages and considered as one of the main methods of proper mathematical investigation. What constitutes rigor has changed over the years: modern mathematics returned to Euclidean geometry, revealing missing axioms and finding gaps in proofs, while trying at the same time to reaffirm its consistency together with the consistency of the 19th century analytic geometry. Nevertheless, the basic tools and methods of Euclidean geometry persisted throughout the centuries: an infinite line, a circle and a scribe – a system of basic signs and propositions – from which every other true proposition can be derived.

It is at this point that Fuller attacks Euclid's geometry, by criticizing its tools: "Euclid limited himself in his theorems to construction and proof by the use of three tools –

1 Cf. Heath 1921, 319; Proclus 1992, 53.

straightedge, dividers, and scribe. He, however, employed a fourth tool without accrediting it – this was the surface upon which he inscribed his diagrammatic constructions.”²

In his paper from January 1944, Fuller presents his position, in which he sets the groundwork for his “energetic geometry” that would later become “synergetics.” Fuller follows up his critique, of the lack of use of a tool that was completely forgotten, by giving a historical explanation:

“It must be remembered that Euclid argued his geometric cases at a time in history when the spherical concept of the universe, which some assert was known to ancient Greek philosophers, had if so, been lost again. At that time, the savants were subscribing to a flat or planar earth concept. Therefore, it is not surprising that his use of that flat plane as a surface upon which to work went as axiomatic. Logical to the misconception was the beginning of his proofs in the special abstract realm of an imaginary plane geometry.”

Fuller’s critique is in effect a contrarian stance against Euclid, whom he accuses of being the one who “had come in by the wrong entrance” and hence had insufficiently reflected upon his own tools. This has led, according to Fuller, to an illusory elementarism in the sciences: it not only reduced geometry into a sequence of logical steps, from which one could eventually draw a conclusion, but at the same time expelled from geometry the pivotal concept of movement,³ at best reducing it to a secondary concept derived from more fundamental objects, which could be removed at any point from the geometrical structure of which it stems. Euclidean geometry, according to Fuller’s conception, is static; the concept of movement is invoked through axioms, a step that can be avoided and is in fact redundant. Fuller says so explicitly, when he remarks:

“We find experimentally that two lines cannot go through the same point at the same time. One can cross over or be superimposed upon

2 This citation and the following two are taken from a 1944 paper by Fuller: *Dymaxion comprehensive system, introducing energetic geometry*. In: Krausse/Lichtenstein 2001, 160–168, here: 164.

3 The history of the mathematical geometrical use of the notions of motion and movement (for example, whether they should be used as tools in mathematical proofs, how they should be conceptualized, what kind of entities – curves, surfaces – do moving objects create) starts already in antiquity; it is intricate and subtle. Aristotle condemned the use of motion in Geometry, stating “[t]he objects of mathematics are without motion” (Aristotle 1928–1952, vol. 8, 989b), whereas Euclid does use the concept of motion in some of his definitions (Book XI of Euclid’s *Elements*, definitions 14, 18 and 21. See Heath 1908b, 261–262). For overviews concerning motion, space and geometry, see e.g. Rosenfeld 1988, esp. chapter 3 and De Risi 2015. As we merely aim to point at the mathematical background against which Fuller developed his own thought, we by no means attempt to give even a partial account of it, as it would take us outside the scope of this paper.

another. Both Euclidian and non-Euclidian geometries misassume that a plurality of lines can go through the same point at the same time. But we find experimentally that two or more lines cannot physically go through the same point at the same time.”⁴

All known geometries presuppose the totality of all lines already exists, since only then can two lines pass through a single point *at the same time*. Fuller makes the claim geometry does not take into account the dimension of time, and therefore may also not take into account time-consuming movement required in order to draw a line.⁵ The movement, which acts as the dynamic aspect of the structure, is in effect what keeps a built structure stable, as we will see in Section 3.1 in connection with *Semper*. According to Fuller, this is not apparent as long as one restricts oneself to plane geometry:

“[...] the Greek geometers were first preoccupied with only plane geometry. They were also either ignorant of – or deliberately overlooked – the systematically associative minimal complex of *inter-self-stabilizing forces* (vectors) operative in structuring any system (let alone our planet) and of the corresponding cosmic forces (vectors) acting locally upon a structural system. These forces must be locally coped with to insure the local system’s structural integrity [...]”⁶

It is clear Fuller’s critique did not merely target Euclidean geometry as embedded in its context of origin. It was rather aimed at its revival during the late 19th century. It is here that we should take a step back in order to understand the mathematical landscape that served background to his critique. What was the conception of geometry during the end of the 19th century to the beginning of the 20th century? How were the concepts of motion and movement reshaped?

2.2 The structural understanding of geometry at the beginning of the 20th century

In this section we will briefly review the conception of geometry from the end of the 19th century until the middle of the 20th century, focusing on Felix Klein’s *Erlangen program* and David Hilbert’s *Grundlagen der Geometrie*, and finishing with Alfred Tarski’s axiomatization of geometry. We wish to highlight that Fuller’s critique did not solely take aim at Euclid’s *Elements*; it was particularly interested in the revival of interest in axiomatic methods. At the end of the 19th century the interest in the foundations of geometry was growing

4 Fuller 1975a, section 517.03.

5 Hence, there is only a partial overlapping of events. See Section 3.2.

6 Ibid, section 986.042.

both from a group-theoretic viewpoint and an axiomatic viewpoint. The emergence of non-Euclidean geometry at the beginning of the 19th century (Bolyai's and Lobachevsky's treatises), the mathematization of space via Riemannian manifolds and the mathematical definition of curvature prompted major philosophical questions regarding the nature of space and its epistemology.⁷ The emergence of non-intuitive geometries gave rise to a need to discover the relations between the axioms of geometry and experience. In order to give a proper albeit incomplete historical account of this period, we begin with a recourse to *group theory*, which was one of the main topics of mathematical investigation during the 19th century, and which served as one important source for the development of a conception of geometry of that time.

A *group*, denoted by the letter '*G*', is set of elements equipped with a binary action, denoted by '***', which fulfills certain requirements. An obvious example for a group is the set of whole numbers together with addition as its binary action. The requirements the action should fulfill are considered to be the most elementary, when we think about actions such as addition or multiplication. To be more specific, there are four requirements: *closure* (if the elements *a, b* belong to *G*, denoted as $a, b \in G$, then $a * b$ belongs to *G*, denoted as $a * b \in G$), *associativity* (if $a, b, c \in G$, then $a * (b * c) = (a * b) * c$), *unit element* (there exists an element $e \in G$ s.t. $e * g = g * e = g$ for every element $g \in G$) and *inverse element* (for every $g \in G$ there exists an $h \in G$ such that $g * h = h * g = e$).⁸

The study of group theory and its applications is usually considered to originate from the work of Évariste Galois (1811–1832), who was working on the necessary conditions for solving an algebraic equation using the four known arithmetical operations (addition, subtraction, multiplication and division) together with roots of any order. What interested Galois around 1830 was not the equations themselves or their solutions, nor was he interested in the type of algebraic relations the roots hold among themselves. He was interested instead in the set of permutations of the roots themselves that preserve their algebraic relations.⁹ In other words, Galois's discoveries prompted a process by which numbers were no longer considered fundamental to algebra. More crucial was a grasp of the algebraic-structural setting for which numbers assembled into various sets serve only an example and considered as a derivative of this structure.

7 For an extensive survey on the changing face of geometry during the 19th century see Gray 2006.

8 This definition can be found in all textbooks on group theory. See e.g. Rotman 1999, 12.

9 For example, for the equation $x^4 - 5x^2 + 6 = 0$, the solutions are $A = \sqrt{2}$, $B = -\sqrt{2}$, $C = \sqrt{3}$, $D = -\sqrt{3}$ and one of their mutual relations is: $AB + CD = -5$. Not every permutation of the roots *A*, *B*, *C* and *D* will preserve this relation. For example, if the permutation, denoted by *f*, is $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow D$, $D \rightarrow A$ then $f(A)f(B) + f(C)f(D) = BC + DA = -2\sqrt{6} \neq -5$. More surprisingly, out of the set of 24 possible permutations of 4 elements, only 4 permutations preserve the above relation.

The concept of a *permutation group* was derived from developments in the theory of algebraic equations and from what became known as Galois theory. This historical strand is just one of the roots of group theory. Indeed, Klein's Erlangen program makes it clear the development of the concept of the abstract group had another historical root, namely, geometry. Felix Klein (1849–1925) was a German mathematician and mathematics educator, known for his work in group theory and non-Euclidean geometry, and for his work on the connections between geometry and group theory that spurred his Erlangen program.¹⁰

Klein's program incorporated the idea that to every geometrical entity one can associate an underlying group of symmetries. By *symmetry* we mean a one-to-one transformation of the space onto itself that preserves certain properties of the space in question. The notion of a group is essential here: its set of elements was the set of symmetries, and the binary action was composition, as in the composition of functions. If S is our space (e.g. S is the three-dimensional Euclidean space), and f is a symmetry transformation of S (e.g. f acts by rotation with respect to an axis) then there are distinct subsets of S , which are not transformed by f (e.g. the axis of rotation). From this standpoint, Klein stated the task of geometry as follows:

“Given a manifold and a group of transformations of the manifold, to study the manifold configurations with respect to those features, which are not altered by the transformations of the group.”¹¹

The mathematical hierarchy of geometries is thus represented as a hierarchy of these groups, and the hierarchy of their invariants. For example, lengths, angles and areas are preserved with respect to the Euclidean group of two-dimensional symmetries, while only incidence and cross-ratio are preserved under the more general group of two-dimensional *projective transformations*. One might be under the impression that, in opposition to Fuller's conception of the Euclidean *Elements*, Klein's Erlangen program does indeed deal with movements and transformations (such as rotation, translation and reflection). However, let us consider the following citation from Klein's: “*We peel off the mathematically inessential physical image and see in space only an extended manifold; [...] transformations of manifold [...] also form groups*”.¹² Together with peeling off the “inessential physical image”, one obtains a removal of any physical movement at the foundation of geometry. In this respect, Fuller might have regarded Klein's program as a

¹⁰ Klein 1872.

¹¹ Klein 1893, 67.

¹² Ibid.

descendant of the axiomatic method: group theory deals with movement as an abstract movement that can and should be formalized and axiomatized; a static structure, that is.¹³

One consequence of Klein's program was that it enabled the acceptance of Hilbert's axiomatic-structural approach to geometry. Indeed, as Wussing states "the transition to the notion of an abstract group was a partial cause, as well as a partial effect, for the growing acceptance of the 'axiomatic method' in Hilbert's sense of the term."¹⁴

Recognized as one of the most influential mathematicians of the late 19th and early 20th centuries, David Hilbert (1863–1943) was a German mathematician, who advanced research on the axiomatization of geometry, culminating in one of his most influential works: *Grundlagen der Geometrie*. It should be noted that Hilbert was not the first to suggest geometry should return to its axiomatic origins. Moritz Pasch, Mario Pieri and Hermann Wiener,¹⁵ among others, also dealt with the subject at that time. However, Hilbert's approach was decisive for the way geometry was conceived in the early 20th century. Hilbert conceived of geometry as a natural science, one in which intuition plays a crucial role, though its experimental foundations may be regarded somewhat retroactively.¹⁶ Hilbert states in his lectures on mechanics:

"Geometry is an experimental science [...]. But its experimental foundations are so irrefutably and so generally acknowledged, they have been confirmed to such a degree, that no further proof of them is deemed necessary. Moreover, all that is needed *is to derive these foundations from a minimal set of independent axioms* and thus to construct the whole edifice of geometry by purely logical means."¹⁷

Once a minimal set of independent axioms is put together, geometry is studied through logical means:

"Geometry [...] requires for its logical development only a small number of simple, fundamental principles. [...] [T]he choice of the axioms and the investigation of their relations to one another is [...] tantamount to the logical analysis of our intuition of space. The following investigation

13 See Wussing 1984, Part III.2 for an extensive analysis of Klein's program, and 194–196 for a description of mechanical movements in terms of group-theoretic concepts. It should be noted that Klein was also an ardent supporter of the use of models in mathematical teaching and research especially in the field of geometry. See for example: Mehrtens 2004; Sattelmacher 2013; Rowe 2013.

14 Wussing 1984, 251.

15 Pasch 1882; Wiener 1892; Pieri 1898.

16 See Corry 2004, chapter 3.

17 Ibid, 162. See also Corry 1997.

is a new attempt to choose for geometry a simple and complete set of independent axioms [...]”¹⁸

Hilbert’s views on geometry in particular and mathematics in general therefore did not regard mathematics as an empty formal game;¹⁹ they rather emphasized independence and consistency of an axiomatic system derived from intuition and experience. That view was promoted in *Grundlagen der Geometrie*, where Hilbert’s objective was to identify and fill ‘gaps’ or remove ‘extraneous hypotheses’ in Euclid’s reasoning. The manuscript laid out a clear and precise set of axioms for Euclidean geometry, and demonstrated in detail the relations of those axioms to one another and to some of the fundamental theorems of geometry.

In *Grundlagen der Geometrie* Hilbert considers three collections of basic objects, which he calls ‘points’, ‘straight lines’ and ‘planes’, and five relations between them. The conditions prescribed in Hilbert’s system of axioms are sufficient to characterize the basic objects and their relation to each other. In order to prove axiomatic independence, Hilbert builds several different geometries by negating some axioms while keeping others intact. Albeit possibly counter-intuitive, the resulting geometries are consistent. Geometry’s innate structure is maintained as a consistent one, unrelated to physical reality, to which it does not correspond. This can be seen in Hilbert’s words:

“We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as ‘are situated,’ ‘between,’ ‘parallel,’ ‘congruent,’ ‘continuous,’ etc.”²⁰

What points, lines and planes have are their relations to each other. An object ‘point’ does not necessarily refer to a point in the physical sense: the only necessary and sufficient condition for it to be such is that it satisfies the relations between what is called ‘point’, ‘line’ and ‘plane’. It divorces geometry from any recourse to a specific instinctive meaning (or notions such as movement or motion). This was apparent already in 1893, when Hilbert, upon his return from Halle after hearing Wiener’s lecture, famously said: “One should always be able to say, instead of ‘points, lines, and planes’, ‘tables, chairs, and beer mugs’”.²¹

The understanding that geometry is not about describing a space, but rather about conceiving it as what is grounded in a system of axioms, gave rise to a plurality of different

18 Hilbert 1899, 1.

19 See Corry 2004, 161.

20 Ibid, 2.

21 Blumenthal 1935, 402–3.

geometries. It opened the way to view geometry (and algebra) first and foremost as an internal structure, one that is not based on movement, measuring or counting. This is manifest in the shift from Hilbert to Tarski. Hilbert, having advanced mathematical formalism considerably, still regarded geometry as fundamentally empirical, though experimentation in itself need not be performed.²² Tarski, on the other hand, considered geometry wholly in its structural interiority. Alfred Tarski (1901–1983) was a Polish logician, mathematician and philosopher considered as one of the greatest logicians of the 20th century. He proved in 1930 that geometry, once formulated according to a specific choice of notations and axioms, admits an elimination of quantifiers: every formula is equivalent to a Boolean combination of basic formulae, that is, geometrical propositions can be written using first order logic alone. Once setting up the basic objects, relations and axioms, every claim of Euclidean geometry can be formulated using the quantifiers \exists ('there is') and \forall ('for every') together with its basic objects serving as variables.²³

While Hilbert is considered one of the influencing mathematicians to reformulate to Euclid's axiomatic geometry, it is Tarski who found a more economic and efficient axiomatization for it.²⁴ Tarski's system of axioms for Euclidean geometry was based on a single primitive element – 'point' – and two undefined relations among those elements – *betweenness* and *equidistance* (or congruence). For every three points a , b and c , the relation 'betweenness' takes the value 'true' if the point b lies on the line segment with ends a and c . For two pairs of points – thinking of each pair as the endpoints of a line segment – the relation 'equidistance' holds if the two segments are of equal length. All other relations are consequently derived; for example, the collinearity of three points is defined in terms of betweenness (a , b and c are collinear if and only if one of them is between the other two). Tarski did not take 'line' or 'incidence' to be primitive notions; indeed, the only primitive notion is the point.

The primary significance of Tarski's elementary geometry lies in its satisfying three essential meta-mathematical properties: it is *deductively complete* (every assertion is either provable or refutable), *decidable* (there is a procedure for determining whether or not any given assertion is provable), and it is *consistent* (and this is why it is a correct axiomatization).²⁵ In order to prove these three, Tarski, in a move similar to Hilbert's, based geometry on the real numbers. To prove the completeness of the systems of complex algebra and Euclidean geometry, Tarski proved the completeness of the system

22 Concerning Hilbert's contribution to the rise of modern algebra and modern geometry, see for example: Corry 2004, chapter 3; Mancosu 1998, Part III; Hasse 1932.

23 Here is an example of one claim of Euclidean geometry: for any triangle, the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side.

24 By "more efficient" we mean that Tarski proved with this axiomatization that the euclidean geometry is consistent. For an extensive survey on Tarski's life and work, see Feferman/Feferman Burdman 2004.

25 See Tarski 1967.

of algebra based on real numbers – one that Hilbert assumed as evident and therefore did not bother to prove.²⁶ Not only that, Tarski noted: “it is possible to construct a machine which would provide the solution of every problem in elementary algebra and geometry”.²⁷

This mechanical description of geometry is expressed in Tarski’s formulation: all axioms and propositions are expressed in terms of first order logic. For example, the famous parallel axiom can be expressed as follows:²⁸

$$[B(abf) \wedge ab \equiv bf \wedge B(ade) \wedge ad \equiv de \wedge B(bdc) \wedge bd \equiv dc] \rightarrow bc \equiv fe$$

where the variables are points and $B(-,-,-)$ designates betweenness. This is a description that does not resemble Euclid’s in any form: “If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.”²⁹

In Tarski’s framework one does not need *several* basic objects. Such plurality might induce problematic relations between these objects, or a tacit form of abuse of notation might take place, as seen in Hilbert’s *Grundlagen der Geometrie*.³⁰ A single object is all that is called for – an abstract object without presupposed properties, bearing no particular relation to empirical reality or intuition.³¹ Its properties are exclusively derived from a system of axioms: the point in Tarski’s work is an object defined according to what satisfies the axioms.

What is then the essence of geometry in its various faces from Klein to Tarski? It is clear Fuller’s critique bears merit, though not entirely well grounded from a historical standpoint. From Fuller’s perspective, motion and movement were formulized so that they became pure mathematical objects, a maneuver that leads to a reduction of dynamics into axiomatics, that is, a static structure. Hilbert’s views on geometry encouraged a consolidation of it as what does not have an essential connection to movement (as a line can also be named a chair). Fulfilling Hilbert’s program for an axiomatically consistent geometry, Tarski had come to speak of geometry in mechanical terms. Tarski no longer refers to geometry as the study of space (together with constructions in and through it); he rather refers to its meta-properties as a static structure. Following Fuller, one may

26 See Hilbert 1899, section 9.

27 Tarski 1967, 306.

28 Tarski/Givant 1999, 184, axiom 10₃.

29 Heath 1908a, 155.

30 Note that in *Grundlagen der Geometrie* a line is a collection of points but also functions as a basic object.

31 Cf. Hilbert’s reference to Kant’s citation regarding the origin of abstract ideas from intuition (Hilbert 1899, 1).

say the Greeks' static constructs (e.g. the square or the cube) were replaced by a static structure for geometry itself.

2.3 The two sides of the mathematization of the fold at the 19th century

In light of a static conception of geometry, we ask how folding, as a dynamic operation, was perceived mathematically starting from the 19th century. Before turning to Fuller's conception of the fold, we will shortly examine the dual role folding played in mathematics at that time. This will help us situate Fuller's thought within a pertinent historical tradition.

A folded piece (of paper, fabric etc.) is regarded as such when one or two of the following operations are involved: creasing (as in folding a paper by a mountain- or valley-fold) or bending (without introducing creases). In this section we provide two examples of 19th century mathematizations of folding that took both operations into account: Sundara Row in his 1893 manuscript *Geometrical exercises in paper folding*, and Leonhard Euler, who described developable surfaces as folded. These mathematizations considered folding not only as a mathematical tool, but also as what expresses essential characters of the geometric form.

2.3.1 Row's Folds and the emergence of the physical straight line

Tandalam Sundara Row was an Indian mathematician, who worked for the Indian government in the revenue department. Row is mainly known for his book *Geometrical exercises in paper folding*.³² Klein's favorable mention of Row's work in *Vorlesungen über ausgewählte Fragen der Elementargeometrie* sparked a general interest in the geometry of paper folding.³³ Why was Klein so impressed by Row's work on folding? To answer this question, let us examine how Row deals with geometry. To begin with, Row refers to the folding of paper as "kindergarten gifts" (the word 'Origami' does not feature). He invokes Friedrich Fröbel's gifts and occupations: "[t]he idea of this book was suggested to me by Kindergarten Gift No. VIII. Paper-folding".³⁴ Row states that "[t]hese exercises do not require mathematical instruments," referring to the straightedge and compass used in Euclidean geometry.³⁵ Row also dispenses with the need for axioms:

"The teaching of plane geometry in schools can be made very interesting by the free use of the kindergarten gifts [the paper folding]

32 Cf. Friedman 2016 for a detailed account of Row's life and work.

33 Klein 1897, 42: "[...] we may mention a new and very simple method of effecting certain constructions, paper folding. [...] Sundara Row, of Madras, published a little book *Geometrical Exercises in Paper Folding* [...], in which the same idea is considerably developed."

34 Row 1893, vii.

35 Ibid.

would give them [school children] neat and accurate figures, and impress the truth of the propositions forcibly on their minds. It would not be necessary to take any statement on trust.”³⁶

Row suggests teaching Euclidean geometry to children could be done without axioms, that is, without “statement[s] [taken] on trust.” In comparison to Euclid, Row proposes a different conception of geometry: a geometry not grounded in axioms or ideal objects, but rather based on folding as its one and only allowable operation. As a result, the status of the straight line, as a product and producer at the same time, becomes clearer.

The opening chapter to Row’s manuscript starts with a description of materiality, not with any foundational system of axioms:

“Look at the irregularly shaped piece of paper [...] and at this page which is rectangular. Let us try and shape the former paper like the latter. Place the irregularly shaped piece of paper upon the table, and fold it flat upon itself. Let X’X be the crease thus formed. *It is straight.*”³⁷

Row starts with an operation based on paper and hence on materiality: the folding of an “irregularly shaped” sheet of paper and later the passing of a knife.³⁸ The important point to consider here is that the line produced is straight as a direct result of folding.³⁹ There is no need to prove the line is straight, or define it as what passes through two points.

As the line X’X is only considered a consequence of folding, it obtains another status: it is that along which we fold: “Fold the paper again as before along BY, so that the edge X’X is doubled upon itself.”⁴⁰ Row now folds the paper along the line that was just created, such that a part of this line X’X will be folded upon itself. When considering the crease BY that is created, Row discovers that BY and X’X are perpendicular.⁴¹ Creating thus a rectangle, Row continues with the folding of a square whose side is of unit length. Then a smaller square is folded inside, rotated by 45 degrees in relation to the initial square. The process continues repetitively, creating via folding a sequence of squares embedded one into the other.

In Row’s treatment, the straight physical line acquires a special status: it is at once created by the fold and creating it. It is crucial to emphasize Row always deals with line

36 Ibid, viii.

37 Ibid, 1 (our italics).

38 Ibid.

39 In contrast to Kempe’s 1887 treatment of straight lines (Kempe 1887, 2–3).

40 Row 1893, 1.

41 “Unfolding the paper, we see that the crease BY is at right angles to the edge X’X.” (ibid).

segments – the inevitable result of folding a piece of paper of finite dimensions. There are neither infinite lines (and hence no dispute over the parallel axiom), nor basic objects to begin with. There is rather a basic operation that initiates geometry. The basic objects, the *Grundbegriffe* and the relations between them do not play the same crucial role in Row's book, as they did for many of his contemporaries. Row takes into account neither group theory nor axiomatic methods he was surely well aware of.⁴² In Row's work it is the fold – as what causes the *discrete*, finite, straight line to emerge as a material, *discrete* unit – that plays the crucial role.

2.3.2 Euler, folded surfaces and differential geometry

Let us now turn to developable surfaces: in this context the fold is considered a continuous operation. The history of developable surfaces can be traced as far back as Aristotle (384–322 B.C.).⁴³ In their current definition, developable surfaces are regarded as a special type of ruled surfaces: they have zero Gaussian curvature and can be mapped onto the plane without distorting curves.⁴⁴ Though the history of developable surfaces deserves a detailed account, we will only provide a brief survey focusing on their relation to folding.⁴⁵ In his development of calculus, Leonhard Euler (1707–1783) initiated the first serious mathematical study of ruled surfaces. He wrote his celebrated manuscript *About solids, the surfaces of which can be developed on the plane* – in the original: *De solidis quorum superficiem in planum explicare licet* – where he identified surfaces as boundaries of solids. Euler opened the manuscript with the statement that cylinders and cones have the property that they can be flattened out or “developed on the plane” unlike spheres. Euler wished to know which other surfaces share this property.⁴⁶

It is important to note for the purpose of our discussion that *explicare* in Latin means ‘to explain’, ‘to develop’ but also ‘to unfold’. The expression “in planum explicare,” which features all throughout the paper,⁴⁷ can be translated verbatim into ‘to unfold onto a plane’. The term ‘developable surfaces’ is a later nomenclature.

Euler failed to find developable surfaces (besides cylinders and cones) through analytical means. Using geometric principles, however, he did reach a solution. Employing geometrical results, Euler understands that lines that were parallel on the flat paper will also not meet on the folded one, concluding that the line element of the surface has

42 Row's awareness of other mathematical methods can be seen in Row 1906. Note the same year (1893) another manuscript on folding was published by the mathematician Hermann Wiener. See: Friedman 2016.

43 Aristotle states in *De Anima* that “a line by its motion produces a surface” (Aristotle 1928–1952, vol. 3, 409a).

44 Gaussian curvature is defined as the product of the two principal curvatures, which are the eigenvalues of the second fundamental form of the surface in question (the second fundamental form being a quadratic form defined on the tangent plane to a point on the surface). See e.g. Pressely 2001, 147.

45 For more detailed surveys see: Cajori 1929; Reich 2007; Lawrence 2011.

46 In Euler's words, “quorum superficiem itidem in planum explicare licet.” In: Euler 1772, 3.

47 Ibid, 7, 8, 11, 27, 31 and 34.

to be the same as the line element of the plane. What is surprising perhaps is that the geometric principles in question were inspired by folded paper: “charta plicae”.⁴⁸ It was folded paper and not solids, which informed the intuition behind developable surfaces in their early incarnation.⁴⁹

Euler was not the only one to employ such terminology: the mathematician Gaspard Monge (1746–1818) also studied developable surfaces at the time, and, as with the former, described developable surface (and curves on them) as *pliée*, i.e., ‘folded’.⁵⁰ It might be claimed that mathematicians (e.g. Monge, Euler) considered folding during those decades an essential action for creating surfaces, as an operation grounded in the materiality of the paper. However, it is important to remark that, with the further development of calculus and the rise of differential geometry, the term *Manifold* (*Mannigfaltigkeit*), albeit having an etymological connection to ‘fold’, was not chosen to describe surfaces as inherently *folded*. In his 1854 talk *Über die Hypothesen, welche der Geometrie zu Grunde liegen*,⁵¹ Bernhard Riemann used the term *Mannigfaltigkeit* almost synonymously with ‘magnitude’, when he stated he set himself “the task of constructing the notion of a multiply extended magnitude,”⁵² and invoked various motivations when first using the term. ‘Mannigfaltigkeit’ for Riemann can equally be discrete; it does not necessarily refer to a surface. When talking about continuous manifolds, the intuitions Riemann provides for choosing the term “Mannigfaltigkeit” are positions of objects and colors. No wonder a developable surface was and is considered a manifold and not a folded piece of paper.

3. Fuller’s mobile structures

As was seen in sections 2.2 and 2.3, a withdrawal from materiality occurred in geometry at the end of the 19th century: consider for example Tarski’s obvious mechanization of geometry. Row’s manuscript on the other hand was either completely ignored or criticized for being “too infantile for a grown person.”⁵³ Against this background, Fuller suggested that stable geometry (in the form of planes and lines) emerges in fact from mobile moving folds, threads and transformations.

48 Ibid, 7.

49 Euler was of course also one of the founding fathers of topology, along Henri Poincaré, Solomon Lefschetz and Johann Listing. Fuller was interested in topological transformations (e.g. the Jitterbug transformation, see section 3.5) and was aware of Euler’s polyhedron formula: $V - E + F = 2$ (see section 3.4).

50 For example in: *Mémoire sur les développées, les rayons de courbure, et les différens genres d'inflexions des courbes a double courbure* (Monge 1785, 517–519); *Application de l'analyse a la géométrie, a l'usage de l'Ecole impériale polytechnique* (Monge 1809, 348 among others), *Géographie descriptive* (Monge 1811, 141).

51 Riemann 1868. Cf. also Cantor 1878, where it can be said that both mathematicians took manifolds as sets.

52 Riemann 1868, 133: “Ich habe mir daher zunächst die Aufgabe gestellt, den Begriff einer mehrfach ausgedehnten Größe aus allgemeinen Größenbegriffen zu construieren.”

53 Young/Young 1905, vii.

3.1 Fuller and Semper: folds and interlaces

Folds and folding are not the primary consideration in Fuller's work. However, the most characteristic of his artifacts – should they be experimental buildings, maps or geometric modeling – are indeed folded or otherwise rely on folding as a deforming operation, as can be seen in fig. 1.⁵⁴ Considering how vitally important folds and folding were for Fuller's practical design, his remarks on the issue were dispensed sparingly, with most dedicated to specific problems of folding, such as the great circles.⁵⁵ Where one would otherwise expect a theory of folding to accompany Fuller's rich discourse on design, it is only found implicitly in his artifacts and the geometry of the Synergetics.⁵⁶ This disproportionality calls upon us to rediscover a tacit theoretical foundation from which to reconstruct the fold and the deforming operation.

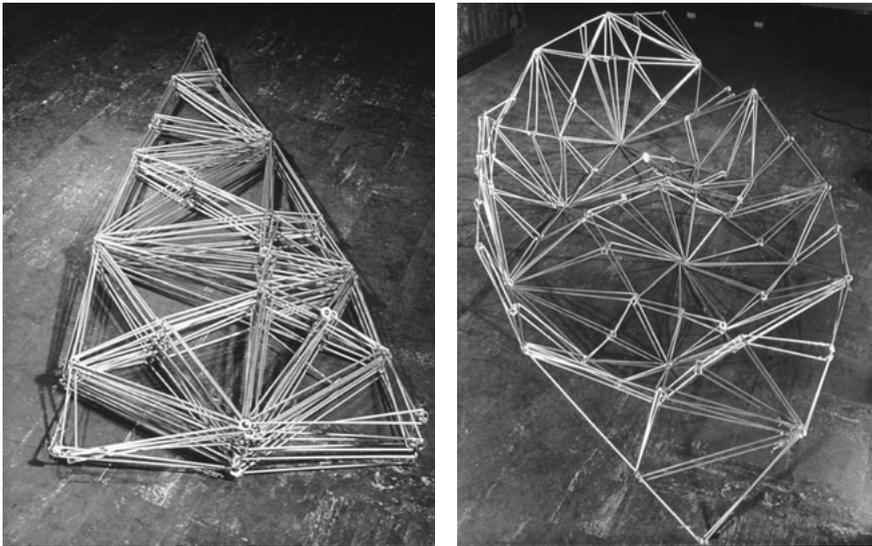


Fig. 1: Necklace-Dome: One of the first folded geodesic domes of Fuller, done in 1950.

54 See also fig. 6 and 7: the *Jitterbug transformation*.

55 Fuller 1975a (sections 450–9) demonstrates eight models (a cuboctahedron and an octahedron, among others) that can be constructed by folding whole circles (with a protractor, using origami-style folding). See Fuller 1975a, section 459.03: "The six great circles of the icosahedron can be folded from central angles of 36 degrees each to form six pentagonal bow ties." Cf. also Fearnley 2009.

56 The changing and developing relationship between theory and design in Fuller's work is seen in: Krause/Lichtenstein 1999; Krause/Lichtenstein 2001.

In his earlier studies Fuller examined ways to reduce weight loads in architecture and construction. He was famous for provoking fellow architects with the question: “Does anybody know what a given building weighs?”⁵⁷ Weight load reduction, employed as a design strategy, was for him a means for examining economical and efficient construction. Fuller noted how weight specifications come up naturally in the design of marine vessels, vehicles and aircrafts, while in building construction this information is considered irrelevant. His question attempted to bridge the gap between the two practices (or mentalities): the mobile and the stationary.

Techniques for consolidation, folding and size or shape adaptation are present in all mobile forms of human habitation (such as tents, yurts and tipis). This is true not only for the architectural structures themselves, but also for equipment and furniture that go along with them. For a nomadic way of life, weight is not the only crucial criterion. Objects belonging to the household must fit requirements for transport. Folding fulfills these requirements in great measure: it allows objects to assume various shapes, being either flattened or spatially expanded. Folding allows for a transformation, with which objects can be adapted to mobility. Folds are thus both a result and an expression of movements, whose event-patterns Fuller summarizes under the concept of precession.⁵⁸

One can observe firsthand the direct link between movement and folding in everyday clothes and textiles: dresses, cloaks, curtains, carpets and so on, as well as adjustable flexible space partitions.⁵⁹ The fact that, under this aspect of regulation between inner and outer, a systemic correspondence between organisms and artifacts can be devised, is not least suggested by the fact that the use of hides of animals and barks of trees belongs to one of the oldest techniques for space subdivision.

The architect Gottfried Semper (1803–1879), regarded as one of the originators of research into material culture, derived his theory of architecture from primitive artifacts, such as clothing, used for space partition. This theory finds expression in his monumental work *Style in the Technical and Tectonic Arts; or Practical Aesthetics* (1860–3).⁶⁰ What is of special

57 “Does anybody know what a given building weighs? I once asked an American Symposium of architects including Raymond Hood and Frank Lloyd Wright as well as the architects of Rockefeller Centre, the Empire State Building and the Chrysler Building what the different structures they were designing weighed. Clearly, weight was not one of their considerations. They didn’t know.” In: Fuller 1963, 53.

58 “The effects of all components of Universe in motion upon any other component in motion is precession, and in as much as all the component patterns of Universe seem to be motion patterns, is whatever degree they affect one another, they are inter-affecting one another precessionally, and they are bringing about resultants other than 180 degrees. Precess means that two or more bodies move in an interrelationship pattern of other than 180 degrees.” In: Fuller 1975a (section 533.01), 287.

59 Fuller encompasses an overarching notion of a dwelling place with “environment controls.” See Fuller 1963, 55 ff; Compare Krausse 2002b, 97 ff.

60 Semper 2004.

interest for us here is his account of the textile origins of architectural space enclosures, exemplified through the wall as an architectural element. In his draft of 1853 he writes:

“We have in our German language a word which signifies the visible part of the wall, we call this part of the wall, *die Wand*, a word which as a common root and is nearly the same with *Gewand* which signifies woven stuff; the constructive part of the wall has another name, we call it *Mauer*. This is very denoting.”⁶¹

“Denoting” hence distinctive. Here one finds not only two classes of materials – fabric and fiber on the one hand, rocks and soil on the other – but also two different principles of structure, reflected in two different types of construction. While hard crystalline materials tend to resist compressive forces till they give way to pressure in the form of fractures and fissures, fiber-based materials absorb tensile, attractive forces and bending stresses; in contrast to crystalline materials they are flexible. Semper shows in his early writings, it is the latter that preceded masonry.

“It is a fact,” he writes, “that the first attempts of industrial art, which have been made and which we still observe to be made by human beings, standing on the sill of civilization, are *dresses* and *mats*. This part of industry is observed to be known even by tribes, which have no idea of *dress*ing.”⁶²

Plaits, carpets, interlaces and hangings were originally used for space arrangement and partition, to which solid structures were subsequently added,

“the thick stone-walls, were only necessary with respect to other secondary considerations, as for instance to give strength, stability, security etc. Where these secondary considerations had no place, there remained the hangings the only means of separation; and even when the first became necessary, they formed only the inner scaffold of the true representative of the walls, namely the variegated hangings and tapestries.”⁶³

Semper demonstrated how these elements enable flexible interior partitions.⁶⁴ Flexibility and mobility originally form a unit that is lost with the use of solid structures, and must

61 Semper 1983, 21.

62 Ibid.

63 Ibid.

64 As an example, Semper cites the Caribbean hut in which the walls are transferable and not connected to the roof. Cf. Semper 1986, 34f.

be compensated for using doors. This idea experienced a modernist revival in the form of mobile room dividers and separators, sliding doors and accordion folding partitions. A most striking example is found in *curtain walls*, whose construction in recessed reinforced concrete columns (as in Bauhaus Dessau 1926) accounts for the old truth: the space-enclosing elements of architecture are in effect suspension structures subject to tension.

3.2 Fuller's non-simultaneous foldings

Semper's reintroduction of fiber-based materials into the processing and manipulation of form was taken on by Fuller; this time of course under the conditions of advanced industrialization, new materials, innovative construction techniques and global transportation systems. Fuller defines the fundamental relationship of human existence to mobility as follows:

“Man was designed with legs – not roots. He is destined to ever-increasing freedom of individually selected motions, articulated in preferred directions, as his spaceship, *Earth*, spinning its equator at 1000 miles per hour, orbits the sun at one million miles per day, as all the while the quadrillions of atomic components of which man is composed inter-gyrate and transform at seven million miles per hour. Both man and universe are indeed complex aggregated of motion.”⁶⁵

This is a concise summary of what Fuller called *scenario universe*. It is this scenario that forms an indispensable part of the exposition to Fuller's energetic-synergetic geometry.⁶⁶ A scenario is favored over theorems or axioms; it emphasizes the a priori temporality of a (geometrical) event:

“The Universe”, so presented in his book *Synergetics*, “can only be thought of competently in terms of a great unending, but finite scenario whose as yet unfilled film-strip is constantly self-regenerative [...]. Our Universe is finite but non-simultaneously conceptual: a moving picture scenario of non-simultaneous and only partially overlapping events”.⁶⁷

The reference to the scenario and to the agility and mobility of the film expresses Fuller's deep mistrust in the image, the still image, the single frame with its implied immobility.

65 Fuller 1969, 348.

66 Fuller 1975a (section 320.01–02), 87.

67 Ibid.

The single image evokes the illusion of simultaneity of events, as in the image of the starry sky, an image that exhibits light, which in fact emanated from stars at different moments in time.

Only when considering longer time spans can one observe evolution and metamorphosis in nature. An emphasis on non-simultaneity (with partial overlapping at the most) is also present in the way ‘folding’ is present in the German words *Überlegung* and *überlegen*. The verb *überlegen* has indeed the following three different meanings: 1) cover; coat; overlap, 2) consider; contemplate, 3) lay an object down over another object.⁶⁸

The introduction of the scenario, as a form of thinking and of presentation, allows us to work with partially overlapping events, where scenarios are both descriptive and prescriptive – prescriptive with respect to actions that would be performed, carried out and so executed.⁶⁹ The performative aspect of the scenario also plays a role in Fuller’s geometry, which insists on the embodiment and the materialization of geometric figures in the model, as well as in the live-performance of transformations that he discovered. How did Fuller come to adopt the scenario as a framework for cognition? We already detect its origins in his first architectural project, as a framework for design. The structure, which he has in mind, is not developed in response to the layout of its designated lot, but in accordance with easy transport. The tower house, which was designed in 1928, could be industrially prefabricated and then shipped by air (with a Zeppelin); it could be delivered to any location on the globe. Fuller, even before clarifying what was needed and implied in constructing such a house, first simulated this unprecedented procedure. To that end, Fuller drew a series of sketches in the manner of a comic strip, which depicted the key events of this scenario.⁷⁰

Fuller’s recordings from this period show how attentively he followed the development of this popular genre and reflects on its potential as a form of presentation: “Undeniably the ‘funnies’ are the most generally inspected portions of our daily newspapers and may be considered the economic frosting that sells the cake – It is more than significant that these funnies have completely lost race of ‘slapstick’ and have become serials of handy philosophy.”⁷¹ Even later on, in his preparation of maps and diagrams of complex global data (world energy map, global transport development, history of isolation of chemical elements), Fuller insisted on “maintaining a comic strip lucidity”.⁷²

68 Grimm 1936, column 385.

69 Regarding the various aspects of performance in Fuller’s work, cf. Krausse 2016.

70 Krausse/Lichtenstein 1999, 99–103.

71 Krausse/Lichtenstein 2001, 102.

72 Ibid, 152 (from Fuller’s *Earth incorporated* (1947)).

Traces of scenario-thinking, partial overlapping and comic-strip lucidity can be found also in Fuller's pictorial depiction of the construction process. Take for example the sequence of photos that illustrated the construction of the *Dymaxion house* starting from its components, through the individual assembling steps, to the finished and furnished residential house. The same pattern appeared in his second attempt to build this house – albeit with an altered outline – in an aircraft factory. This time it was indeed realised as a prototype.⁷³ Besides the process of assembling, the most important thing about this sequence of photos is its demonstration of initial and final construction stages: the initial assembly of lightweight thin parts, occupying but little space, set against the finished space-consuming building. Before construction commences, building parts are laid out as one spreads clothes before packing a suitcase.

In this way one may inspect all components in order; they were designed to fit into a container in the most space efficient way. In the case of the *Wichita house* of 1946, the cylindrical, metal, storage container served also a key structural element of the building. Packaging aligns well with the concept and practice of folding. Transportation to and unpacking at the construction site need to be taken into account in the design of the container and its contents. Unpacking should fit color-coded step-by-step assembly all the way up to the finished, fully furnished, turn-key house. This turns building into a performance that follows a precise scenario.⁷⁴

It is no coincidence, that this performance, as in a sequence of movie frames, resembles the process of the unfolding of a plant from seed to bud to leaf, save that its origin goes back to design, from which mechanical parts are developed as affiliated and connected.

3.3 Seedpods, Viruses and Geodesic domes

Fuller related design scenarios to organic growth processes on various occasions. One of his experimental constructions, the *Flying Seedpod* of 1953, is a pure folding mechanism.⁷⁵ Flying Seedpod is a dome, 42 feet in diameter, which can set itself up semi-automatically. Whether compacted as a transportable bundle or deployed as an architectural structure,

73 The corresponding photo series of the Dymaxion House and the Wichita dwelling machine are printed in: Marks 1960, 84 f and 128–133.

74 Fuller's scenario-thinking looks beyond the finished product onto its ultimate end-of-use. With its structures he envisages "dismountability", with its materials, "recirculation." Responsibility in design extends to the entire life cycle of the product – what Fuller called "cradle-to-grave." It took the combination of product-cycle together with recycling to go from "cradle-to-grave" to the slogan "cradle-to-cradle". Regarding "dismountability" cf. Marks 1960, 112–113. Regarding recycling, cf. Fuller 1938, chapter 38, 316–322; Reprint in: Krause/Lichtenstein 2001, 117–120. Regarding "cradle-to-cradle", cf. Braungart/McDonough 2009.

75 See fig. 2: series of four photos "Flying Seedpod" 1954/5.

all parts stay connected to each other by flexible nodes or joints. The bundle consists of 30 inwardly folded tripods, whose chains come together in ball joints. The system of tripods can be unfolded and straightened up by extending pistons in pneumatic cylinders – radially directed tubes at the vertices of the dome. With the extension of pistons, a net made of cables is stretched. The clear span unsupported dome-structure obtains firmness and rigidity, through the interaction of its push-pull components. Flying Seedpod was a project that Fuller realized in 1953 with students from Washington University (fig. 2).



Fig. 2: Flying Seedpod. Washington University, St. Louis, 1953; A folding-out geodesic structure.

The study of folding structures of geodesic domes developed alongside progress in space exploration missions, so that one might see in Flying Seedpods – “the first scientifically designed apartment” – a rocket capsule to the moon.⁷⁶

Though the seedpod was nicknamed “the moon structure,” it did not fly to the moon. Instead it appeared in other ways in the world of molecular biology. Fuller tells how it came about:

76 “You may possibly be looking at the prototype of the structural principles that we may use in sending history’s first (little) scientific dwelling to the moon. As you see, all the structural members are tightly bundles together in parallel so that they may be transported in minimum volume within a rocket capsule.” In: Fuller 1965, 70; Fuller’s foundations for folding structures were later continued by his pupil Joe Clinton for NASA. Cf. Clinton 1971.

“The principle of structural dynamics of the [...] moon structure, the flying seedpod and its logistic pattern transformability, are double interesting because they have turned out to be also the same structural, self-realization system employed by a class of microcosmic structures – the protein shells of all the different types of viruses. About three and one half years ago molecular biologists in England and their colleagues in America, working in teams, were trying to discover the structural characteristics of the protein shells of the viruses with X-ray diffraction photographic analysis. These virus scientists discovered that the viruses’ protein shells were all some type of spherical geodesic structure. Having previously seen published pictures of my geodesic structures they corresponded with me and I was able to give them the mathematics and show them how and why these structures occur and behave as they do. They have now found the poliovirus structure to be the same structure as the possible ‘moon structure’. The polio virus instead of having the tripods on the outside and the clusters of five and six feet on the inside, has the five – and six – way jointings outside and the tripods or three–ways on the inside.”⁷⁷

The encounter between Fuller’s experimental architectural structures and science of the day could be passed for incidental – a random correspondence between structures on widely disparate scales – if not for geometry that provides a connection of a more general nature.

The researchers at Birkbeck College in London engaged with a striking resemblance between viral capsids and Fuller’s geodesic domes,⁷⁸ when the largest was just completed, almost 120 m in diameter, making it the largest ever built clear span dome.⁷⁹ It appeared as if the same such spherical structure found in nature was anticipated by Fuller, or rather as if he had built his geodesic domes according to models from nature. The first recorded images of capsids produced by an electron microscope were published in 1962; they finally made resemblance evident.⁸⁰ On this basis and other of Fuller’s 1960

77 Fuller 1965, 72.

78 Cf. Morgan 2003, 86: “In the mid 1950s, Francis Crick and James Watson attempted to explain the structure of spherical viruses. [...] biophysical and electron micrographic data suggested that many viruses had > 60 subunits. Drawing inspiration from [Fuller’s geodesic domes and] architecture, Donald Caspar and Aaron Klug [...] proposed that spherical viruses were structured like miniature geodesic domes,” by forming a (protein) shell. Indeed, “[t]he idea was that identical viral subunits could bind together in quasi-equivalent positions to form a shell with > 60 subunits while conserving the same specific contact pattern between subunits” (ibid, 88).

79 Union Tankcar Company Geodesicdomes, Baton Rouge, Louisiana; Railway repair facility, October 1958, cf. Marks 1960, 222f.

80 Ubell 1962, 1.

architectural structures, the mathematician Harold Scott MacDonald Coxeter was able to match individual domes, each slightly different in its geometric resolution of geodetic networks, to individual types of viruses, whose capsids likewise vary geometrically.⁸¹ This new visual input, brought to bear through advances in electron microscopy (EM), made the resemblance even more apparent in microorganisms. Fuller got to see electron microscope images of marine microorganisms (magnified up to about 50,000 times), taken by biologist Gerhard Helmcke (who specialized in lightweight constructions in nature), in a 1962 meeting together with his colleague Frei Otto and Helmcke himself. The architect Frei Otto, who had just launched the research group *Biology and Building*, later reported how impressed Fuller was:

“The stereoscopic photographs looked like models of [Fuller’s] famous domes. To the participants it was clear: Had he [Fuller] known the diatom shells before, the whole world would have said that he had learnt this by watching the living nature. Had he knew the diatom shells, how they were really, he would not have probably dared to build his shells.”⁸²

When Fuller met Aaron Klug and his interdisciplinary team of researchers in July 1959 there were no clear images yet, only clues coming from the X-Ray analysis of crystals. Even when Klug succeeded in applying crystallographic EM to the analysis of complex viral capsids, the images obtained were rather confusing: due to its extensive depth of focus, all structures were depicted one on top the other. Klug, who received the Nobel Prize in Chemistry in 1982 for this work and others, remarked in his Nobel lecture: “Thus, we knew what we were looking for, but we soon found that we did not understand what we were looking at”.⁸³ Fuller’s geodesic domes served not only as a possible guide to deciphering those EM images. They were geometrical models at large which allowed pattern recognition of an unidentified micro-phenomenon.

3.4 Platonic solids: a stable habitat?

The point of contact between Fuller’s designs and structural chemistry is derived from a functional correspondence: both deal with a problem of habitation that needs to be solved with utmost care for resources – one might consider protein shells to be the smallest houses in nature. In the case of the virus the space within is occupied by DNA and RNA molecules, which are densely folded and packed waiting for a suitable host to open the

81 Coxeter 1971. On the relationship between Coxeter and Fuller, see Roberts 2006, chapter 9.

82 Otto 1985, 8.

83 Klug 1992, 89.

protein shell. Nevertheless, both Fuller's domes and capsids represent an attempt to solve an optimization problem – maximum capacity for the smallest surface area. The geometric solution to the problem is the sphere, which both Fuller and viruses opt for.⁸⁴ Turning now to the structural elements of an approximately spherical casing or shell, a method of regular subdivision of the sphere must be developed.

In subdividing the sphere, Fuller – similarly to viruses, marine microorganisms and carbon molecules – chooses another way, compared with dividing the globe into a network of longitudes and latitudes. He avoids the classical construction of a dome using meridian groins and horizontal rings or bands. Instead, the sphere is symmetrically divided into regular polygons, described by Plato as the Elements in his dialog *Timaeus*.

Common to all Platonic solids – the tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron – is that their vertices lie on a circumscribed sphere and their edges, once projected onto the circumscribed sphere, form arc segments of equal length. The arc segments are all part of great circles. Roughly speaking, great circles are the paths of minimal length on the sphere. They correspond to the straight lines of Euclidean geometry. Together, great circles and lines belong to a class of paths known as geodesics. Back to Platonic solids, the system of geodesic segments, obtained through projection onto the circumscribed sphere, forms a regular grid that divides the surface of the ball into equal polygons.

With the icosahedron, one obtains the most tightly arranged subdivision; it consists of 20 equilateral triangles adjoining along 30 edges and touching at 12 vertices. According to Euler's characteristic formula, $V - E + F = 2$, where V denotes the number of vertices, E the number and F the number of faces. Five edges meet at the vertices of the icosahedron. A fivefold rotation symmetry is maintained throughout all of its subdivisions. This becomes clear when one looks at the truncated icosahedron:⁸⁵ the 12 vertices are trimmed; one third of each edge is truncated at each of both ends, resulting in a new polyhedron consisting of 12 pentagons and 20 hexagons, with each pentagon surrounded by five regular hexagons. The truncation of the icosahedron affords a way to refine the spherical subdivision thereby better approximating a sphere. The truncated icosahedron is now well known for the Telstar football and the discovery of the carbon molecule C_{60} . The 1996 chemistry Nobel laureates, Harold Kroto, Richard Smalley and Robert Curl, responsible for the discovery, named it *Buckminsterfullerene* in recognition of the architect's work.⁸⁶

84 We leave aside a class of rod-shaped viruses such as the prototypical tobacco mosaic virus.

85 Comprising 32 faces, 90 edges and 60 vertices.

86 Kroto 1996. See also Krause 2002a.

3.5 From folding and foldable platonic solids to the Jitterbug transformation

A question that occupied virologists was whether viral capsids – comprising at least 120 protein subunits under seemingly identical conditions – contrived their structure in a manner similar to Fuller’s geodesic domes, domes which might have more than 180 of triangular subunits. These domes are able to modify themselves slightly, such that they arrange themselves according to a geodesic grid on a sphere.⁸⁷ Fuller could clarify this through his grid and through a formula, describing the total number of edges of these triangulated domes. Situating at every corner of a triangle a ball of radius 1, Fuller imagined a growing shell structure, beginning with the 12 balls, whose center situated on the surface of a sphere, then while the structure grows and another layer is situated symmetrically on the outside of the former layer, there are 42 balls, then 92 balls and so on, according to the formula: $n = 10f^2 + 2$. With this formula, Fuller bases his calculation on the *cuboctahedron*, being one of the 14 semi-regular Archimedean polyhedra: while inscribing a cuboctahedron with edge length 1 in a sphere, Fuller instructs, as explained above, to posit 12 balls centered on the vertices. As one may enlarge the sphere and the cuboctahedron (when now the edge length of the cuboctahedron is 2), one may posit 42 balls centered on the vertices, edges and faces of the cuboctahedron. In Fuller’s formula n stands for the balls situated symmetrically in the growing shells, where f stands for the number of the layers of these growing structures.⁸⁸ And the higher the frequency – that is, the number of layers, the greater the number of balls and hence triangular subunits (created by drawing a line between three adjacent, tangent balls) is and the more fine-meshed the network of geodesic structures. Fuller adds that “[t]hese successive layers, which permeate each other in all directions may be identified with energy waves radiant in all directions from a nucleus.”⁸⁹ By “nucleus” Fuller points to the fact, that while situating the 12 balls of radius 1 on the sphere, having their center as the vertices of the cuboctahedron, there is a room for an additional ball – called “nucleus”, located exactly in the center of the sphere and touches all the other 12 balls (see fig. 5).

Fuller’s mathematical model was therefore not developed from the icosahedron or from any of the other Platonic solids: as we will explain later, when balls are positioned at the vertices of the icosahedron, the structure thus obtained does not have a nucleus, from which “energy waves” emanate. To see how a nucleus is necessary, one constructs

87 “[...] we have discovered that the way these viruses were built was similar to the way the geodesic domes were Built. Geometrically you cannot put more than 60 identical units on the surface of a sphere with each one making identical contacts [...] The virus we had been working on had 180 sub-units – three times 60 – so they couldn’t all be in identical environments.” In: Klug 1995, 10.

88 For a detailed explanation, see: Edmondson 1987, 114–116. For a proof of Fuller’s statement: Coxeter 1974.

89 Krause/Lichtenstein 2001, 169.

dense *sphere packing* using identical spheres. By connecting the centers of spheres in certain arrangements one can obtain the elementary forms, for example Platonic solids.

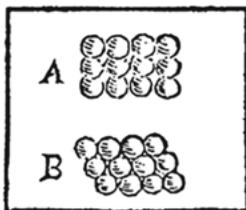


Fig. 3: Kepler's two types of planar packing of spheres.

Sphere-packing arrangements – being an arrangement of non-overlapping, possibly touching spheres – go back to Johannes Kepler's 1611 book *De Nive sexangula*. In it Kepler distinguishes between two types of packing: one that forms a cube and another that forms a tetrahedron.⁹⁰ Kepler's book is primarily an investigation into the hexagonal form of snowflakes. As to the densest space-filling arrangement of spheres, Kepler conjectured that the tightest packing produces rhomboidal aggregates, known as the *rhombic dodecahedral honeycomb*. The density η of a packing of solid spheres is today defined as:

$$\eta = \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{\infty} \mu(K_i \cap B_t)}{\mu(B_t)}$$

where $\mu(X)$ is the volume of X , B_t is a ball of radius t centered at the origin, and K_i are balls which are used for the packing.⁹¹ It follows that the density of a packing of balls is always smaller than 1. When Kepler discusses sphere packing, he proposes two types. The first is the simple cubic packing and the second is what is called today an *FCC packing*, i.e. the hexagonal arrangement. On the cubic arrangement, Kepler concludes: "The arrangement will be cubic, and the pellets, when subjected to pressure, will become cubes. But this will not be the tightest pack." However, when considering the second packing, Kepler remarks that "[t]his arrangement will be more comparable to the octahedron and pyramid. The packing will be the tightest possible, so that in no other arrangement could more pellets be stuffed into the same container".⁹² It is known today that the density of the FCC packing is $\pi/(3\sqrt{2}) \sim 0.7405$, whereas the density of the simple cubic

90 See: Kepler 1966. See also fig. 3.

91 See e.g. Conway/Sloane 2013, 8.

92 Kepler [1611] 1966, 15.

packing is approximately 0.6802, but Kepler does not give any reason why this pyramidal arrangement is the densest.⁹³

Fuller studied sphere packing through his work on the suspended construction of the Dymaxion House. For this purpose, he used cables wrapped around an inner rope. When cross-sectioning the cables, one sees that the cross-section consists accordingly of six units around one center (“six around one”. See fig. 4b).⁹⁴ In this cross-sectioning Fuller sees a prototype of symmetrical growth. Fuller’s first study of a wave-mechanical matrix appears in grids for his hexagonal layout of the house on a mast, in which the intervals are specified not only in length dimensions, but also in time units.⁹⁵ The initial intuition for the *matrix* (or *isotropic vector matrix*) is found in the hexagonal configuration packed with circles: it belongs to the class of two-dimensional densest packing. Connecting the centers of adjacent circles with lines, one obtains a part of a configuration of 9 triangles, which the Pythagoreans called *tetractys*, as can be seen in fig. 4a.

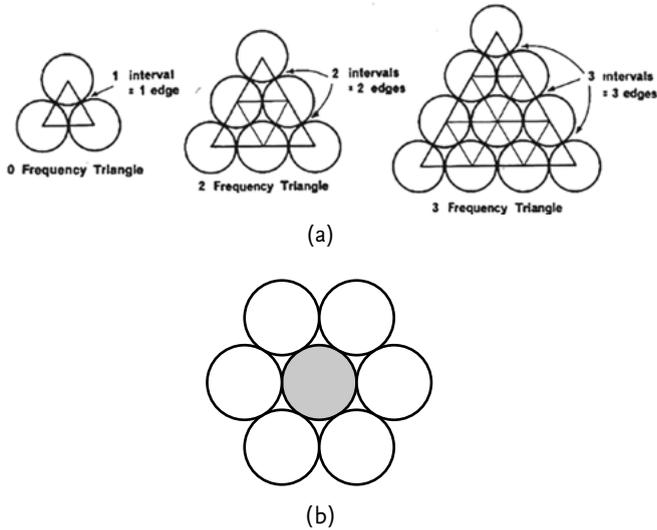


Fig. 4: (a) Only with the most right image (the *tetractys*) one may notice the appearance of the nucleus, in the middle of the inscribed hexagon. (b) Fuller’s “six around one” construction, obtained from the tetractys. The nucleus is the grey circle.

93 Kepler’s assertion, better known as the Kepler’s conjecture (that the densest packing of identical balls in space is either a cubic arrangement or an hexagonal arrangement), was proved only in 2014 by Thomas Hales.
 94 In the magazine he edited SHELTER (November 1932, 106–107) Fuller assembled pictures that, inter alia, show snow crystals, cable cross-sections and the Dymaxion House hexagonal plan. See facsimile reprint in: Krause/Lichtenstein 1999, 172–173.
 95 Krause/Lichtenstein 1999, 114 f. “Time based plan for the 4D House”, Figure 1928.

When extending this method to three-dimensional densest packing modeled on the cuboctahedron, Fuller indicates:

“When the centers of equiradius spheres in closest packing are joined with lines [i.e. a possibly infinite truss modeled as the cuboctahedron], an *isotropic vector matrix* is formed. This constitutes an array of equilateral triangles which is seen as the comprehensive coordination frame of reference of nature’s most economical, most comfortable structural interrelationships employing 60-degree association and disassociation.”⁹⁶

What Fuller searched for becomes clear in his 1938 book *Nine Chains to the Moon*. There Fuller calls for a time-based geometry that takes into account propagation of waves and rays and growth processes in space and time: “Time, or how far (or more properly ‘fast’) radially outward, in time and space, integrated as rate or the center of the sphere.”⁹⁷ The mental image Fuller uses is that of the “expanding sphere” or the “halo” – a radiation in all directions. As was seen above, Fuller finds a framework for such processes in a polyhedron consisting of 8 triangles and 6 squares: the *cuboctahedron*. In its representation using sphere packing there are not “six around one” but rather “twelve around one.”⁹⁸ Its shell consists of 12 balls. Growing symmetrically with additional layers, 42, 92, 162, 362 (or more) spheres can be packed. The cuboctahedron may be regarded a form of sphere packing: it forms a shell of 12 balls with a nucleus. Having a different shape compared with that of the cuboctahedron, one might say the icosahedron consists merely of a shell, whereas the cuboctahedron has its nucleus, as noted above and as can be seen in fig. 5.

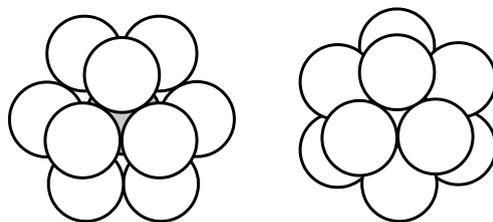


Fig. 5: On the left, a packing of 12 white balls, where the center of each (white) ball is placed on a vertex of a cuboctahedron; note the existence of a grey nucleus. On the right, a packing of 12 balls, where the center of each ball is placed on a vertex of the icosahedron; in this case there is no space for a nucleus of the same size. Note that the density of the icosahedral packing is approx. 0.6882, being lower than the density of the FCC packing.

96 Fuller 1975a, caption to figure 420.02 (our italics).

97 Fuller 1938, 134.

98 Fuller 1975a, 116–120, section 413.00.

Since the hollowed space in the center of the icosahedron has smaller dimensions compared with the ball in the nucleus of the cuboctahedron, the radii of the balls layering the icosahedron are somewhat shorter than the edges. Therefore, the icosahedron cannot provide a basis – a matrix, in Fuller’s terminology – for symmetrical growth processes. It always remains the same perfect space division, through all of its modifications: be it a capsid, a shell or a geodesic dome. Conversely, the cuboctahedron provides a matrix for growth processes – what Fuller refers to by “isotropic vector matrix” – but is unsuitable as a building structure. The six square faces of the cuboctahedron lack stabilizing diagonals. Fuller has repeatedly demonstrated this effect, for example, in his *Necklace Performance*.⁹⁹ A square, according to Fuller, is not a structure; it is a temporary opening at the most. Only the triangle is a structure, as it is self-stabilizing. This feature appears only when one builds the cuboctahedron as a model and connects the rods with flexible joints.

These and many other aspects come to light in a geometric transformation Fuller named *Jitterbug* after a 1940 popular dance. Though there are different ways to dance the Jitterbug,¹⁰⁰ we will limit ourselves to one, the easiest, which is performed with a model. The model consists of 24 individual rods that may move while being connected to flexible nodes. Thus, various configurations can be produced via folding. It begins with the most extended configuration: the cuboctahedron. Though the cuboctahedron also has square faces, we focus on the behavior of the triangles during the transformations (see fig. 6 and 7). Its 14 faces can be seen clearly, and it is evident via touching that the cuboctahedron is not rigid. When slight pressure is applied to the model’s upper triangle, the result is a left or right rotation of the remaining triangles. This draws the 6 squares in a diagonal direction, deforming them into rhombi. At this point the squares crease, forming an invisible, “silent” edge, while an inserted rod could have prevented this deformation. When the rods are inserted in the middle of the creased squares, the result is of an icosahedron: each folded rhombus supplies 2 triangles. A total of 12 new triangles plus the original 8 add up to the 20 triangles of the icosahedron. In the absence of intervention to stop the process of folding it goes on: adjacent edges of the cuboctahedron’s original squares join in pairs. The result is an octahedron, whose 8 triangles are formed with double-tipped edges. The next two stages of the Jitterbug transformation are more complicated since they require further extension and folding of area, suffice it to say, this procedure produces a tetrahedron, with quadruple edges and culminates in a planar triangle with eightfold edges. Throughout the process of transformation the polyhedra emerge in a process of phase transitions.

99 Fuller 1975a, 317–319, sections 608.00–609.01.

100 Cf. “Five ways to dance the jitterbug” in Krausse/Lichtenstein 2001, 24–33.

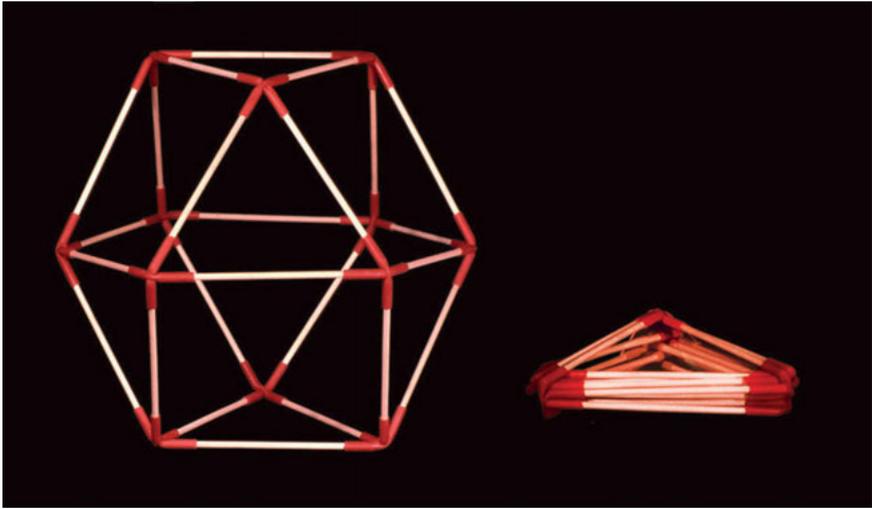


Fig. 6: The initial and the final position of the Jitterbug transformation cuboctahedron and triangle.

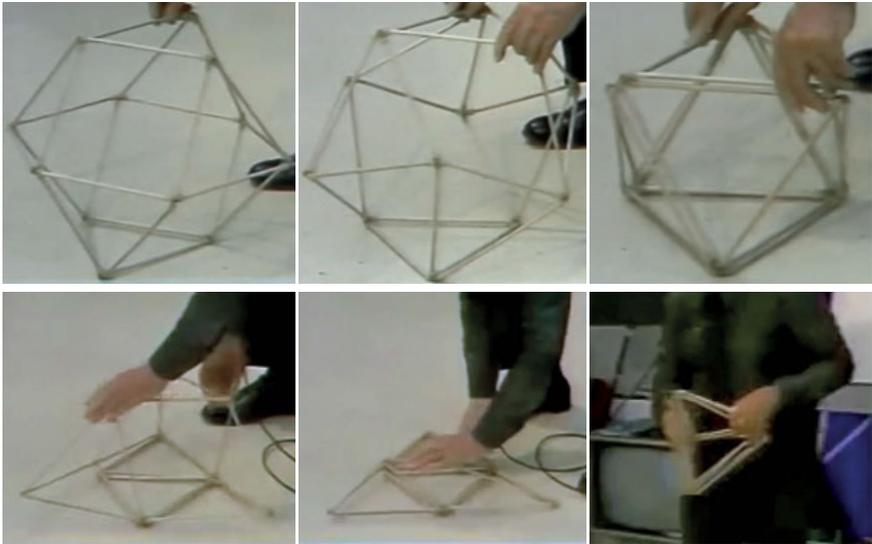


Fig. 7: Photos taken from R. Buckminster Fuller's 1975 lecture "The Vector Equilibrium", where several stages of the Jitterbug transformation are presented.

The Jitterbug transformation demonstrates convertibility in a rigid and stable structure; it demonstrates metamorphosis from one "solid" into the next in a single, continuous process of contraction and expansion up to the structure's limits. This metamorphosis not only offers a different view on geometry, it also offers another perspective on thought: equally abundant in forms, the Jitterbug transformation is also abundant in patterns of movement. Fuller demonstrates ongoing articulations of movement have a relationship to epistemology:

“There is quite possible a scientific truth to be evolved from the fact that motion, particularly rhythmic motion, is highly provocative of thought objectivation. Certainly travel provides perspective, broad angles, and accelerated progression potential of clarification of the experience trend.”¹⁰¹

4. Folding: The provocation of thought

Mobile structures, as with the Jitterbug transformation, are at the core of Fuller’s thought. Opposing geometry’s withdrawal from materiality and architecture’s withdrawal from mobility, Fuller suggests a revival of both concepts back into geometry. Indeed, *partial* overlapping and non-simultaneity are pilled off via the axiomatized mechanized conception of geometry – as all lines and axioms appear at once – whereas the mathematization of the fold ignores its materiality as a guiding principle. Following Semper and taking the German word *Überlegung* as a cue, folding suggests an interlacing of thought and contemplation together with a materialized geometry and partially overlapping events. This enables Fuller to tie together his conception of geometry as a material mathematical science – a point is a place where two lines pass through but not at the same time – and the scenario as a form of thinking. Folding, weaving and interlacing engender this form of thinking. As Semper indicated, it is flexible, mobile, foldable interior partitions that enable the wall to be a pure structural element, not the other way around. Fuller takes on this viewpoint and pursues it into the macroscopic world (seedpods and the dance of human beings) as well as into the microscopic world (inspiring the discovery of the structure of viral capsids through his geodesic domes). It is indeed the same line of thought that is apparent in Fuller’s work: geometry, whose essence is exemplified in the folding and unfolding of platonic solids as they metamorphose through the Jitterbug transformation, is not a rigid structure that lacks movement or consideration to materiality, but rather it is the thought-provoking *scenario universe* of material, flexible, non-simultaneous, partial overlapping.

101 Fuller 1938, 139.

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