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Meysam Joulaian, M. Sc.,  
Hamburg

## The hierarchical finite cell method for problems in structural mechanics



# The hierarchical finite cell method for problems in structural mechanics

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The finite cell method (FCM) is a combination of the fictitious domain approach and high-order finite elements. This thesis is concerned with the study of the numerical challenges of this method, and it investigates possible approaches to overcome them. Herein, we will introduce and study different numerical integration schemes, such as the adaptive integration method and the moment fitting approach. To improve the convergence behavior of the FCM for problems with heterogeneous material, we will also propose two high-order enrichment strategies based on the hp-d approach and the partition of unity method. Moreover, the application of the FCM will be extended to the simulation of wave propagation problems, employing spectral elements and a novel mass lumping technique.

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# Preface

This thesis is the result of a research project, funded by the Deutsche Forschungsgemeinschaft (DFG), that I carried out between June 2011 – Feb. 2016 at the institute of Ship Structural Design and Analysis (M-10) at Hamburg University of Technology (TUHH). During this memorable time I had the pleasure to meet and closely work with several awesome fellows.

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My list continues with my colleagues at M-10 institute, my research mates at TU München, and my partners at OVGU Magdeburg. In order to make reading their names more enjoyable, I made a word search puzzle that includes all their first names<sup>1</sup>. Have fun finding them!

Q	B	K	S	H	S	M	K	O	L	M	N
A	F	N	O	A	O	L	L	T	U	L	H
A	L	R	A	H	S	Z	I	S	C	C	S
T	S	I	A	H	S	C	I	N	I	A	I
T	I	M	R	A	P	L	H	R	A	E	M
S	E	N	L	E	V	E	L	A	N	N	E
D	A	U	O	A	Z	U	T	K	O	R	O
X	L	G	N	S	R	A	L	S	V	A	N
P	I	K	C	I	R	T	A	P	P	J	N
N	A	F	E	T	S	O	M	I	D	B	D
S	K	A	C	H	R	I	S	T	I	A	N
J	U	T	T	A	M	A	R	C	E	L	D

At last but not at least, my heartfelt thanks goes to my family and friends for their moral support and blessings. In particular, I am indebted to my beloved wife, without whose love neither life nor work would bring fulfillment. Dear Sahar, your companionship certainly made this journey enjoyable, unforgettable, and more fun. Thank you so much!

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To Sahar  
and to my family  
for their love, endless support and encouragement!

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## Notation

The notation and operators that are used throughout this thesis are defined here. Note the definitions are based on the Cartesian coordinate system with base vectors  $\vec{e}_i$ ,  $i = 1, 2, 3$ . All the operators and variables will be defined when they appear for the first time.

### Tensors

$A, a$	Scalar value
$\vec{a} = a_i \vec{e}_i$	First order tensor (vector)
$\underline{\underline{S}} = S_{ij} \vec{e}_i \otimes \vec{e}_j$	Second order tensor
$\underline{\underline{\underline{C}}} = C_{ijkl} \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l$	Fourth order tensors

### Matrices

$\mathbf{a}$	Local element single column matrix
$\mathbf{a}$	Global single column matrix
$\mathbf{A}$	Local element matrix
$\mathbf{A}$	Global matrix

### Mathematical tensor operations

$\vec{u} \otimes \vec{v} = u_i v_j \vec{e}_i \otimes \vec{e}_j$	Dyadic product
$\underline{\underline{S}} \cdot \underline{\underline{F}} = S_{ij} F_{ij}$	Inner, scalar or dot product
$\underline{\underline{\underline{S}}} \underline{\underline{\underline{F}}} = S_{ij} F_{jk} \vec{e}_i \otimes \vec{e}_k$	Tensor product
$\underline{\underline{\underline{S}}}^T = S_{ij} \vec{e}_j \otimes \vec{e}_i$	Transposed tensor
$\underline{\underline{\underline{S}}}^{-1}$	Inverse of a tensor
$\det \underline{\underline{S}}$	Determinant of a tensor
$\text{div}$	Divergence operator
$\text{grad}$	Gradient operator

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## Abstract

The finite cell method (FCM) is a combination of the fictitious domain approach and high-order finite elements. Thanks to the fictitious domain approach, the task of the mesh generation in the FCM is drastically simplified as compared to the standard finite element method, where boundary-fitted meshes have to be employed. Moreover, due to applying high-order approaches, with the FCM it is possible to obtain high convergence rates similar to those of high-order finite element methods. These two main characteristics make the FCM a viable tool for the numerical analysis of problems in solid mechanics where the mesh generation is the main bottleneck of the simulation – for instance regarding structures consisting of highly heterogeneous materials, foam-like materials, sandwich plates, or composites which may exhibit debonding, delamination or fiber breakage due to a loading. The FCM's interesting properties do however come with some numerical challenges. This thesis is concerned with the study of some of these challenges, and it investigates possible approaches to overcome them.

The first challenge that is addressed in this thesis is the task of performing numerical integration. In the scope of the FCM, we commonly have to compute **integrals with discontinuous integrands**. Such integrals, unfortunately, cannot be accurately computed with standard quadrature rules. To overcome this issue, we will introduce and study different numerical integration schemes, particularly the *adaptive integration method* and the *moment fitting approach*. We will discuss the algorithms and characteristics of each of these approaches and show that the proposed methods enable us to efficiently and reliably compute the corresponding integrals for 1D, 2D and 3D problems. The second concern of this thesis is the **local enrichment in the context of the FCM**. The local enrichment is required for problems including discontinuities or singularities, for which a degradation of the convergence rate of the FCM is to be expected. An example for such a situation is the case of a problem that involves material interfaces, which is one of our focal points in this thesis. To avoid such drawbacks, we will propose two high-order enrichment strategies based on the *hp-d approach* and the *partition of unity method*. Based on several numerical examples, the proposed local enrichment strategies will be examined in 1D, 2D, and 3D in order to point out the advantages and disadvantages of each method. We will show that if the local enrichment is performed properly in the FCM, it is possible to obtain an accurate representation of the displacements and stresses and to retain the high convergence rate of the method. Finally, the application of the finite cell method will be extended to the **simulation of wave propagation problems**. To this end, we will propose a novel approach based on the combination of the FCM and spectral elements. Here, the main focus will be on the issue of the *mass lumping* when the fictitious domain method is applied as well as on the aspect of efficiently employing an explicit time-integration algorithm such as, for instance, the central difference method. We will show that the proposed approach, which is referred to as the *spectral cell method*, offers a very fast and novel technique with a high convergence rate for the simulation of wave propagation problems of structures obeying complicated geometries.

## Zusammenfassung

Die Finite-Cell-Methode (FCM) basiert auf einer Kombination der Fictitious-Domain-Methode mit finiten Elementen hoher Ordnung. Im Vergleich zur Finite-Elemente-Methode, welche oberflächen-angepasste Netze erfordert, wird die Netzgenerierung durch die Verwendung eines fiktiven Gebiets erheblich vereinfacht. Des Weiteren ermöglicht die Verwendung von Ansätzen hoher Ordnung hohe Konvergenzraten, ähnlich denen der Finite-Elemente-Methode hoher Ordnung. Aufgrund dieser beiden Hauptmerkmale ist die FCM als eine effiziente Methode für die numerische Analyse von Problemen im Bereich der Festkörpermechanik anzusehen, bei denen die Netzgenerierung die größte Herausforderung für die Simulation darstellt. Ein Beispiel hierfür sind Probleme mit stark heterogenen Materialien, schaumartige Materialien sowie Sandwichplatten oder Verbundwerkstoffe, bei denen Belastungen zu Delamination oder Faserbrüchen führen können. Die vorteilhaften Eigenschaften der FCM bringen allerdings auch numerische Herausforderungen mit sich. Im Rahmen dieser Arbeit werden einige dieser Herausforderungen näher erläutert sowie verschiedene Lösungsansätze vorgestellt.

Zuerst wird dabei auf die numerische Integration eingegangen. Die wesentliche Herausforderung ist hierbei die Berechnung von **Integralen mit diskontinuierlichem Integrand**. Solche Integrale lassen sich üblicherweise nicht effizient mit herkömmlichen Quadratur-Regeln berechnen. Um dieses Problem zu lösen, werden in diesem Zusammenhang verschiedene numerische Integrationsverfahren vorgestellt und untersucht, wobei vor allem auf die *adaptive Gauß-Quadratur* und die *Moment-Fitting-Methode* eingegangen wird. Die Algorithmen und Eigenschaften der einzelnen Ansätze werden diskutiert, um zu zeigen, dass die vorgestellten Methoden es ermöglichen, die entsprechenden Integrale für 1D-, 2D- und 3D-Probleme effizient und zuverlässig zu berechnen. Der zweite Schwerpunkt dieser Arbeit liegt auf der **lokalen Anreicherung in der FCM**. Die lokale Anreicherung ist in Problemstellungen erforderlich, die Diskontinuitäten oder Singularitäten beinhalten, die in der FCM die Konvergenzrate reduzieren. Ein Beispiel hierfür sind Probleme, bei denen Materialgrenzflächen auftreten, was gleichsam einer der Schwerpunkte dieser Arbeit ist. Um diese Herausforderung zu lösen, werden zwei Anreicherungsstrategien höherer Ordnung vorgestellt, welche auf dem *hp-d Ansatz* sowie der *Partition-of-Unity-Methode* basieren. Anhand verschiedener numerischer Beispiele werden die vorgeschlagenen lokalen Anreicherungsstrategien in 1D, 2D und 3D untersucht, um die Vor- und Nachteile der einzelnen Methoden aufzuzeigen. Es wird gezeigt, dass durch eine geeignete lokale Anreicherung eine genaue Berechnung der Verschiebungen und Spannungen ermöglicht wird, und somit die hohe Konvergenzrate erhalten bleibt. Abschließend wird die Finite-Cell-Methode auf die **Simulation von Wellenausbreitungsproblemen** angewendet. Zu diesem Zweck wird ein neuer Ansatz vorgestellt, der auf einer Kombination der FCM mit spektralen Elementen basiert. Hierbei liegt der Schwerpunkt auf dem *Mass-Lumping* unter Verwendung der Fictitious-Domain-Methode sowie auf dem effizienten Einsatz eines expliziten Zeitintegrationsalgorithmus, wie z.B. des zentralen Differenzverfahrens. Es wird gezeigt, dass der vorgestellte Ansatz, die *Spectral-Cell-Methode*, eine sehr schnelle und innovativ Methode ist, die bei der Simulation von Wellenausbreitungsproblemen in geometrisch komplexen Strukturen zu hohen Konvergenzraten führt.