Pointless Topology
Figuring Space in Computation and Cognition

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Abstract: *The topological turn in computing is elaborated both methodologically and philosophically via a treatment of inference proceeding from the notion of homotopy. In such a program, computation is unmoored from strictly discrete operations and opens up to geometric, or spatial domains, effectively suturing computational processes with procedures of situational embedding. Via this inferential scheme, hypothesis-generation as well as non-trivial transits between singular neighborhoods of thought can be systematically accounted for. The authors have adopted a dialogue format that echoes their presentation and structure of collaboration, while demonstrating a mode of encounter between discrete domains of knowledge.*

Introduction

The topological turn in computing over the last decade can be identified with the research program known as *univalent foundations* (Awodey 2014), an ambitious project that compels a re-evaluation of the core tenets of computational theory. Spearheaded by the work of Vladimir Voevodsky, univalence reorients computation around a form of structuralism rooted in geometry; this presents a treatment of types as continuous maps, forging invariances known as *homotopies* between spaces. The questions it raises reach into the depths of foundational mathematics, demanding a careful examination of what Lautman (2011) considers the key dialectic at the heart of mathematical practice – namely, the reciprocity of the discrete and the continuous, which we can couch in terms of algebra and geometry; the realm of symbols on the one hand and of spaces on the other.

Topology advances the view that all space comes with an attendant structure – that is to say, it is a means of gleaning structure in space. As a fork in mathematical intuition, it marks the move from naive to critical treatments of spatiality. In Euclidean geometry, a space precedes its axiomatization – points, lines and planes assume their own embedding in a surrounding domain which is extrinsic to the axioms provided; space is given in the first instance. Euler undermined the concept of the Euclidean plane with the development of graph theory, while it was Riemann's great contribution to show that there are many species of space, each with its own notion of locality, its own "shape", so to speak. In this new non-Euclidean world, structure is not absolute, but rather context sensitive to its embedding space. As Châtelet would note, this development marked "the liberation of geometry, 'freed' at last from the physical universe" (2000: 6), reframing the relation between algebraic structure and a geometric a priori.

Posing a dialectical relation between the two, as Lautman (2011) does, sidesteps the question of precedence, presenting the challenge of synthesis via what Lautman calls *mixtures*. The arc of post-war mathematics represents a major research program to fuse these two *ur*-branches of mathematical practice, which we can refer to at a high level as the project of algebraic geometry, most notably expressed in category theory and its various strands. This relativistic framework posits a plurality of spaces, each embodying its own logic, preceded by their articulation as mathematical structures such as *toposes*. The body of research associated with topos theory (Goldblatt 2014) presents a novel vocabulary for encoding space in terms of *schemes*, *sections*, *neighborhoods* and *sites*, objects whose algebraic and geometric properties are indistinguishable – providing a toolkit for treating space as a derived mathematical notion.

> From an epistemological perspective, there are notable ramifications to draw from this topological turn concerning possible systematic mobility through various knowledge 'sites', more conventionally called disciplines. Deploying mathematical or physics-based concepts at the level of sheer metaphor or terminological/analogical transfer brings the risk of triviality that works to undermine otherwise important efforts for transdisciplinary scheme-building. When faced with the methodological challenges posed by transdisciplinarity, where the discovery or invention of non-trivial movement through sites of knowledge is crucial, the construction of rigorous modalities of transfer becomes essential. We may think of this, provisionally, as topological learning, which is then also bound

to the question of topological pedagogies. Broadly stated, topology provides a conceptual scaffold for thinking about relations between singularity and unity in systematic ways. Methodologically, topological learning affords the preservation of specificity belonging to discrete neighborhoods of knowledge, while also inventing applied transits across distinct knowledge types.

In Olivia Caramello's theory of topos-theoretic 'bridges', the constructability of space is contingent on what she calls "dynamic unification" (Caramello 2016). Dynamic unification is not a generalization, but occurs where entities are related through a constructed third 'bridge object' that enables the transfer of information, not unlike Fernando Zalamea's evocation of a 'glueing procedure' when addressing mathematical sheaves (2012: 285). Put in Simondonian terms, the bridge object, or glueing procedure, is what enables transduction, the process driving the possibility of individuation. It is what allows for non-trivial comparisons and transfers between discrete entities without sacrificing what is specific to those entities. The guarantee of non-trivial transfers between discrete entities is enabled through Morita-equivalence, which mathematically speaking can preserve syntactic difference while mapping a common "semantical core" (Caramello 2010: 14).

For Caramello, this bridge object, otherwise known as an invariant, is a constructed representation of such a common semantic core between distinct entities. Applying such a topological model to the problems of disciplinary epistemologies in the kind of high-dimensional information contexts under consideration here serves as a fruitful heuristic for overcoming what Lewis Gordon calls "disciplinary decadence" (Gordon 2014). Gordon warns against the turning inward of disciplines that results in both the failure to recognize their limits and a fortification of their self-referential enclosures, secured through the sheer learning of codes and rehearsal of methods belonging to a field. Disciplinary sites calibrate not only what we think (content) but how we think (method), and when petrified they enact "an absolute conception of disciplinary life" (Gordon 2014). The petrification of a discipline is spatially isomorphic with the givenness of its neighborho od enclosure for thought – in other words, the notion of an embedding space 'free' from interference. While a lack of interference may be idealized as smooth and efficient, at the level of human cognition it is also the very condition that disables transfers and thereby prohibits transduction, resulting in a false conflation of metastability and fixity.

Sites and locales

This emphasis on the constructibility of space, which we can trace through topology and category theory, stems from an insistence that a modal structure, an inferential lattice of sorts, accompanies any spatial articulation – what we are referring to as a *site*. Discursive sites are bound not only to phenomenological constraints but to dynamic inferential loci, constraining the possible relations that can manifest in any given embedding space. The apotheosis of this view is the theory of *locales*, which asserts the precedence of an algebraic structure from which a space can be constructed. This is a form of *pointless topology*, outlining a propositional lattice, a kind of inferential frame which is point-free and is not derived from any geometric notion. As the constructive mathematician Andrej Bauer (2013) tries to explain:

> *In the usual conception of geometry, a space is a set of points equipped with extra structure, such as metric or topology. But we can switch to a different view in which the extra structure is primary and points are derived ideal objects. For example, a topological space is not viewed as a set of points with a topology anymore, but rather just the topology, given as an abstract lattice with suitable properties, known as a locale. In constructive mathematics such treatment of the notion of space is much preferred to the usual one.*

Doxastic spaces

Such approaches mark out the topological turn as an insistence on the situatedness of mathematical procedures such as proofs, embedding them in sites conditioned by locales. In this view, there can be no cleaving of the doxastic space of reasons from the mathematized space of geometry, they are both sutured with an inferential structure. The influence of the topological turn on contemporary AI is most markedly expressed by *the manifold hypothesis* (Fefferman/Mitter/Narayanan 2016) – the theory that real-world data can be represented as a complex of manifolds, or continuous surfaces, in an embedding space. Embeddings enable contemporary deep learning models to explore the 'latent space' of relations present in any given form of data, uncovering patterns which cannot be easily distinguished in the input space. This marks out deep neural nets as not simply flat networks of associations, but rather high-dimensional spaces induced by transformations in which the algebraic and the geometric cannot adequately be decoupled. This enmeshing of the continuous and the discrete is another way of stating that an embedding space and its topological structure can only ever exist as a holistic composition. By contrast, earlier models such as *support vector machines* hinged instead on the programmer's ability to construct spaces by explicitly defining their structure up front via *kernel functions;* the models were not able to *learn* embeddings on their own. As such, the field of deep learning can be viewed as a paradigm shift in AI, from the construction of kernels to the induction of embeddings.

A central problem constraining the inferential limits of deep learning is in considering how an embedding space can come to be imbued with a modal structure, which is to say, how possibility can be couched in terms native to a geometric mode of representation. In a sense, how possibility as such can come to be embedded. This question is an expression of a historical theoretical problem in AI, namely *the frame problem*. First articulated by McCarthy & Hayes in the context of robotics in 1969, the problem concerns how a symbolic system can capture the result of actions in an environment, without having to explicitly delineate not just their effects but their non-effects on a variety of other entities. The question relates to the manner in which a situated computational agent negotiates not only its own boundedness but that of all other entities known to it – the means by which it localizes relations and constrains the effects of interaction in its own mental models of the world.

> The effects of interaction, understood as a mode of interference, and its feedback upon mental models of the world are driven by sensitivities to information. By comparison, disciplinary decadence can be described, geometrically, as an absolute fixing of a Euclidean site for certain knowledge types, and through a practice of self-bordering it amounts to conditions for informational desensitization. The degree of transformation of mental models corresponds to the receptivity to re-cognize "signals" or "alerts", as Ramon Amaro and Murad Khan have written in their outline of an expanded picture of interpellation beyond its pejorative guise as that which underwrites the self-transformational opportunity for updating mental models (Amaro/Khan 2020). From a topological perspective, the updating of mental models is akin to recognizing new conditions of situatedness: absent a static or a priori site from which to think, cognitive transformation is equal to the construction of other locales for embedding thought.

> Desensitivity to information is reinforced by an absolute fixing of thought to a given site of embedding. What is lacking is the comparative interference of a bridge object as a creativity-enabling catalyst, effectively serving as the creation of another site from which to interact with a world and its stuff; and it is through such comparative triangulation that abductive cognition or hypothetical reasoning is made possible. In an effort to demystify the type of creativity required for discovery, Lorenzo Magnani emphasizes the role of external representations as a "means to create communicable accounts" of the novel, making them amenable to generative interpellation beyond mere flights of personal imagination, and to the shareable updating of mental models (Magnani 2009: 2). Linking back to Caramello, we may suggest that 'bridge objects' function as just such a necessary external representation, and in some cases as an epistemic mediator for non-trivial transits to novel neighborhoods for embedding speculative cognition. It is via the systematicity of constructing such 'bridge objects' that abductive thought can be justifiably (that is, systematically) performed, via the morphing of spaces for the taking-place of reason as such. As Magnani emphasizes, these transits of abduction do not only operate within theoretical registers of thought, but may also take place via manipulative experiments (that is, environmental interactions with material stuff), introducing an opportunity for transfers between know-of and know-how in hypothesis construction, while broadening the scope of what a 'model' may be. The importance of manipulation in model-based reasoning, particularly vis-à-vis procedures of discovery, is the enablement of a "redistribution of the epistemic and cognitive effort" to contend with entities and information that "cannot be immediately represented or found internally" (Magnani 2004: 233).

The questions raised by interaction, conceptual revision and hypothesis formation pose numerous challenges to a theory of intelligence. The frame problem exposes a key limitation of deductive systems based on classical logic, namely that laws with open-ended sets of exceptions cannot be adequately captured; and the 'common sense' law of inertia, which encapsulates the fact that most actions do not alter most properties of most entities, is just such an open-ended law. For Dennett (1984), the frame problem represents a deep challenge to counterfactual reasoning. Along similar lines, Fodor (2000) considers the frame problem a matter of how we represent modality, which is to say the informational encapsulation of contingency, possibility and necessity. For Brandom

(2010: 79), this is transformed into the question of "doxastic updating", of how an agent updates their beliefs in order to accommodate real-time interaction, and it motivates a position he calls "pragmatic AI", which untethers itself from formal logic in an attempt to address the issue. The Bayesian riposte is to dispense with symbolic reasoning altogether, embracing an inductive scheme in which prior belief is incrementally updated on the basis of feedback – a continuous back-propagation of error fed into a generative model at every instance. This is precisely the strategy taken by deep learning, which evades the question of modality altogether, offering what Pearl (2018) critiques as a "model blind" approach to inference.

In the Bayesian view of intelligence, interference and learning are synonymous, precipitating a fluid updating of prior beliefs. However, rationalist critiques of Humean empiricism are valid here as challenges to the limits of such a scheme, particularly in relation to the epistemological claims made on behalf of deep learning models, which are theorized entirely along inductive lines. In essence, the problem concerns the distinction between prediction and explanation: Brandom (2010) and others reject the supervenience of normativity on empirical generalization, asserting that one cannot possess the affordances of modal reasoning, namely counterfactual robustness in one's inferences, without such capacities. For Brandom, the state of the actual world, its possibility space, and the way it ought to be, are all inextricably enmeshed in everyday acts of reasoning. For Sellars (2007), the account of perception which is central to empiricism is similarly parasitic on a modal structure, which in turn relies on a "space of reasons", whereby accounts of particulars are subsumed by general laws. Lastly, for computer scientist Judea Pearl (2018), model blind approaches to AI are hampered by their rejection of causality, conceived as an objective modal structure of reality. It is the link between the counterfactual and the causal which, in Pearl's view, characterizes the explanatory power of a model *qua* model, as that which singles out a theory as a robust explanation in the first place.

As Badiou (2007) suggests, models are what allow us to think through participation, locating it at the juncture between the sensible (empirically available) and the intelligible (conceptual). Participation is a form of interaction that induces interference as receptivity to stimuli – or, put another way, sensitivity to information. Interference also serves as the critical operator for Turing in his accounts of organized and unorganized machines, in which the latter is linked to the status of an infant cortex. Machines that are random in their construction are what Turing calls "unorganised machines", where interferences gradually set the conditions for machinic operations, a discrete state called a configuration (Turing 1969). The parsing of interference after a certain configuration is achieved produces an organized machine determined for some definitive purpose. Analogously, interference inhibition is isomorphic with doxastic preservation – or, more simply put, conceptual and/or practical habits. What is relevant to highlight from Turing's paper on machinic intelligence is his persistent assertion that so-called "proof" of the impossibility for machinic intelligence can be found in its failures or errors. This is a narrow property and/or expectation of intelligence: not only would every human also fail in this regard, but it undervalues how error is a critical form of interference for reorganizing thought and activity.

On two notions of frame

The challenge of modal reasoning appears irrevocably linked to questions of causality, error and doxastic updating. As such, the frame problem can be cast as a close cousin to the *problem of induction*, but it is not synonymous with the bind presented by Humean skepticism. In a sense, both are intrinsically spatially articulated problems raised by interaction. But the distinction rests on the fact that the former relates to our inferential models of the world as opposed to the causal structure of reality, while neither fully resolves the relationship between the two. Instead, one can say that they both compel a holistic account of reasoning in light of interaction; the prospect of unifying empirical generalization with inferential explanation.

Thus there are two notions of frame one can leverage to approach the frame problem, if one accepts it as a substantial theoretical challenge to computational reason. The first is a modal frame, such as those devised by Kripke (1963). These are foundational in model theoretic treatments of possible world semantics, and are the most common tool employed to resolve this bind. This sense of frame is a kind of meta-language which can be used to create correspondences with a target or object language. It is expressed as a pair <W,R> where W is a set and R is an operation (a binary relation on W). Elements of W are called nodes or worlds. Together the frame and a third component, an evaluation or forcing relation (), are called a model.

The second notion of frame is a generalization of a topological space known as a locale, as previously outlined by Bauer. In this sense a frame is a pointless topology representing a category of open subsets of a space. In this view, frames and locales are both lattices, algebraic forms which describe a topology but precede any geometric expression of a space.

Here we can ask, what kind of frame is the frame problem alluding to? It may be neither or perhaps both, as it is not a strictly mathematical use of the term. In a certain sense, it is a reference to a causal meta-representation one can conceive as a mental model possessed by an agent. In another sense, it is a fundamental challenge to any representational theory of mind. It poses the question of how we can reason about things as other than they actually are, and how we can judge those hypothetical situations to form abductive hypotheses – in effect, to ask which inferential moves are permitted and which are incompatible within our conception of the causal structure of the world, which is ultimately the question of navigation. As a result, the formal treatment of frames one chooses to approach the problem has implications for the semantic analysis of knowledge claims made on behalf of any AI that can be said to deal successfully with the issue it presents.

Pointless topology as a frame for thinking abductive novelty translates into the space of search, such that queries shift from what an object of inquiry is, to where and how it is located/localized (Negarestani 2015). The conditions and contexts of embedding inquiries are open for manipulability, and by transforming the space of search new problems and knowledges can be discovered. It is notably on the activity of constructing problems (more so than identifying 'answers'), that Thomas Kuhn's proposal for paradigmatic scientific change hinges (1962: 37). This model is extrapolated to the social domain by Sylvia Wynter in her account of the necessity to mobilize 'outer views' from which to probe non-adaptation to existing normative spaces for the embedding of thought (Wynter 1984).

There are two ways to consider the navigation of search-spaces outlined by Guerino Mazzola, who distinguishes between receptive and productive navigation – or, in terms we have been otherwise noting, adaptive and non-adaptive (abductive cognition). Receptive navigation is exemplified by an encyclopedia delineating a search-space where knowledge can be augmented, but only within the immutable configuration of its "orientation environment", namely the indexicality of alphabetical ordering (Mazzola 2002: 44). Such a receptive/reproductive mode is optimized for navigating existing sites of knowledge in fragmented ways, but does not permit "less trivial search problems" (Mazzola 2002: 44). Productive navigation, in contrast, changes the very environment or neighborhood of search as a transformation in the conditions of embedding and encoding queries. The dynamism of search-space endemic to productive navigation undoes a more passive idea of navigation as merely the steering of a ship through existing, preconfigured spaces, turning it into the concept of a site of inseparable interaction between objects and conditions. The ongoing challenge inherent to productive navigation, as Mazzola notes, is how to make such novel embeddings of search intelligible to other agents; in other words, how to create such neighborhoods for thought as bridge objects, amenable to participation.

Locales of truth

If the frame problem is a problem about navigation, we can ask what model of cognition gives us the most convincing account of interactive revision. What affordances are required to reason in dynamic environments beyond the brute forcing of a search space? The most comprehensive riposte to the frame problem is broadly connectionist: by eschewing a representational model, one can adopt a form of semantic holism unfettered by symbolic reasoning, one that is closer to a more abstract topological notion, which is an embedding space. This approach addresses many of the doxastic issues which otherwise motivate an appeal to frames, and as such it allows deep learning to sidestep the frame problem entirely. However, this does not totally eliminate the need for inferential structure, as alluded to by Brandom, Sellars and Pearl; it cannot adequately account for normativity, causality or counterfactual reasoning, by appeal to connectionism alone. Instead it casts the frame problem in terms of navigating the modal structure of an embedding space. This directs us toward the second definition of frame, namely a pointless topology, as a more attractive option for approaching the frame problem, by adopting a paradigm which attempts to unify the plasticity of connectionism with the nomological capacity to reason via explanatory models.

It is through the invention of 'props' in either material (manipulable) or conceptual (theoretical) form, serving as mediators of interference, that a reconfigured search-space is rendered sharable, and thus open to partici-

> pation. These are artefacts inviting the abductive intelligibility of unfamiliar neighborhoods for the embedding of thought. For Magnani, abduction must be "an inference permitting the derivations of new hypotheses and beliefs"; as such it is not a new framework that merely explains an older framework, but provides a "very radical new perspective" (Magnani 2009: 32). It is in this way that the locale of a concept is coterminous with the metastable configuration of the vantage points it affords. The proposition here is that the transits between neighborhoods of thought, each belonging to specific conditions of embedding, are dependent upon the construction of bridge objects to pursue non-trivial modes of navigation across concept spaces. A "radically new" site for embedding thought must also privilege modes of access to it, meaning that the possibility of ramifying new search-spaces as a social activity hinges upon the shareable proof of non-triviality, so as to avoid the pitfall of merely constructing proverbial gated communities of thought.

The thread which can be used to suture locales (as frames) with artefactual intelligence is a topological treatment of inference proceeding from univalence. The implications of adopting locales as a guiding paradigm in approaching the frame problem are both methodological and philosophical. The motivation is to ally with the affordances of constructive mathematics, namely computable notions of structure, in order to expand the inferential toolkit available to computational reason, beyond the narrow confines of inductive learning and model blind approaches. This links locales more broadly to the topological turn, as a means of approaching computation from a vantage point which is intrinsically geometric – that is, anchored to sites. The bridging of univalence with the manifold hypothesis thus presents an opportunity for a unified account of computational reason rooted in topology, but it remains a highly speculative proposal. If we wish to gain traction on how rational agents navigate worlds, we should heed challenges such as the frame problem as serious philosophical questions arising from the concretization of models. Such problems provide us with a window onto the thorny and inhospitable landscape sprawling before any contemporary theory of intelligence, hampered as we are at every turn by the embedding of our own cognitive faculties and scientific practices.

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