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Calculemus! The Problem of the Application of Logic and Mathematics*

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Scheibe, E.: **The problem of the application of logic and mathematics.**
Knowl.Org. 23(1996)No.2, p.67-76, 38 refs.
Starting out from Descartes' and Leibniz' idea of a mathesis universalis the achievements of modern mathematics are divided into three major parts: The creation of algorithms, the invention of proofs, and the application of mathematics to the description of nature. This applicability has repeatedly been viewed as being just a miracle. One major idea to diminsh the miraculous impression was to view mathematics as exploring the vast area of all kinds of abstract structures, thus establishing a huge store of humanly possible thinking from which the physicist has only to choose the structure appropriate for the case before him. There remains, however, the problem of mathematical overdetermination of physics: the structures suitable for application usually contain mathematical elements that remain without physical interpretation. The true miracle then seems to be that it is often very difficult, if not impossible, to eliminate those uninterpreted elements from physical theory.
(Author)

1. The Dream of a *mathesis universalis*

As the title of my address indicates, I am going to treat a systematic subject, but in doing so I will not fail to take Leibniz as my point of departure - which is indeed the very least one may expect of an address designated to keynote a convention dedicated to Leibniz. As we all know, the numerous plans entertained - but never completed - by Leibniz included also a plan for a so-called *characteristica universalis* or *lingua generalis*, so let's say: for a universal language with the wonderful properties that its mere grammatical mastery would make one speak truths and nothing but truths, including truths that would be novel ones in a very essential sense. Earlier, Descartes had, under certain conditions, dared

> "to hope for a readily recognizable universal language, easy to pronounce and to write, which, to mention the main point, would also help the human intellect in presenting all objects so clearly to it that it would be well-nigh impossible for it to be deceived (. . .), and by means of which peasants could judge on truth better than philosophers can now"[1].

That was in 1629, and less than half a century later we find Leibniz entertaining similar ideas:

> "If one could find characters or symbols", he says, "which would be capable of expressing all our thoughts as clearly and precisely as arithmetic expresses numbers and analytic geometry expresses lines, then one would evidently be able to do with all objects, insofar as they are subject to rational thinking, that which one does in arithmetic and geometry"[2].

Hence the example after which the universal language of thought is to be patterned is for Leibniz - as it was in a sense for Descartes, too - mathematics, and it is also clear just what it was about mathematics which one hoped to exploit in the new, far more sweeping enterprise: the things one desired to make philosophical capital of were its proofs and its mechanically reproduceable calculations, of whose stringency and simplicity one wished that even the very process of thinking itself should benefit. What blissful state of rationality, once one had accomplished that!

"One would", wrote Leibniz, "convince everyone of one's findings or discoveries, since the calculations could easily be checked out (. . .). And if anyone should doubt my words, I would tell him: 'Let's calculate, Sir!' and, taking pen and ink, we would soon extricate our embarassement"[3].

Leibniz also left us clues as to how he let himself be guided by mathematics in constructing a *characteristica universalis*. The mental germ-cell was some sort of a principle of greater explicitness of language or the reduction of arbitrariness in the symbolic representation of contents. Let us take, for example - to follow Leibniz[4] - the arithmetical fact that three times three equals nine. In the decimal system we express this truth in a form by which no one can tell how this equation came about. The correct formulation of this equation in the decimal system is a mere matter of designation: In the binary system, on the other hand, this question is already disposed of with the first two numbers zero and one, and the representations of the numbers three and nine are already expressions of facts in the binary system. In particular, when calculating in the usual fashion we will obtain together with the product also, in a way, its designation. Correspond-ingly, in the case where non-mathematical and in particular philosophical subjects are included, the intention probably was to construct the universal language in such a way that in its formal structure it would become, to the highest possible extent, an image of the contents of the objects it was designed to express. As Leibniz gushed as late as 1695:

> "If God grants me enough time of life and freedom, I hope to design a kind of philosophy no one has yet seen the likeness of, for it will rightly possess the clarity and certainty of mathematics, containing as it will something similar to calculation. Admittedly, it is not yet possible to decide all

questions with its aid, but such decisions as are taken on this basis are indisputable. (. . .) Once the trail has been blazed, posterity will march forward on it".[5]

Has it so marched forward, and where do we stand today? These are the questions on which I wish to say something in the following - but not, mind you, as a historian, which I am not, but in a reflection by a philosopher of science[6]. In so doing I hope to be able to proceed from the assumption that people like Descartes and Leibniz positively felt that the mathematical disciplines of arithmetic and geometry, already available then as more or less complete, self-contained systems, not only were capable of being developed further intrinsically, but also still fell short of being representative for the entire realm of the mathematically possible in the first place. The development of mathematics in the 16th century was certainly conducive to strengthening such a feeling in any person. The new algebra, the beginnings of analytic geometry and the invention of infinitesimal calculus were clear indications of a beginning expansion of mathematics both in a methodical and an objective respect. It took all the philosophical optimism of the epoch, however, to jump right away to entertaining, and seriously pursuing, the idea of a universal language of thought or a *mathesis universalis*. Even in the present age of giant computers and artificial intelligence we are far removed from imagining that, in the end, *all* rational thinking is - let alone: should be - mathematical thinking. But we can all the more readily sympathize with the expectation of the time that mathematics was about to undergo a major expansion knowing, as we do, with all the undeserved superiority granted by historical hindsight, that that is exactly what happened.

Our reflections in the following will not, however, be restricted to the question of in how far the dreams inspired by the mathematics of the epoch of a *lingua generalis*, and *ars inveniendi*, a *mathesis universalis* have led at least to a new and expanded vision of the mathematically possible.

In the very spirit of the aforementioned classical authors, the concept of the universality of the mathematical includes more than doing justice to the full structural richness in abstracto. It also includes the concrete occurrence of abstract structures in as many fields as possible of reality - for example the far-reaching embodiment of the mathematical in nature. Together with the question "Just what is generally understood by the term mathematics?", Descartes raises the further question "Why not only (arithmetic and geometry), but also astronomy, music, optics, mechanics and several other (branches of science) are designated as mathematical disciplines"[7]. Today one will be the most readily understood if alongside the question of the scope and systematics of mathematics itself one poses the question of its fundamental applicability and the extent of its actual application. Now right here is the point where we have reached the main title of this address, having crossed, as it were, the bridge leading to it from Leibniz's "Calculemus!". The question at issue is how and to what extent the rationalistic claim of the universality of the mathematical presumptuous though it probably was at the time, has meanwhile been honored in theory and practice.

In making a few remarks on this subject in the following, and thus speaking about mathematics and also a little bit about logic, I will be speaking about something which is not everyone's cup of tea. Although everyone will at some point in his or her life have come into contact with mathematics, for many one the upshot of this experience will be no more than the recollection of seemingly endless hours of mathematical lessons at school. Mathematics and logic have entered into everyday language in seemingly different ways. We hear people say that this or that matter is just "higher mathematics" to them, or that some other thing is just "logical", meaning in the first case: "This I don't understand, it is beyond me", and in the second case: "that goes without saying; it is crystal clear". Thus, logic seems to be making out even a little better in popular language than does mathematics. In actual fact, however, what is meant by the second locution is just as little logical in the proper sense as the first one is mathematical in the proper sense. Despite this, on the whole, none too encouraging situation I may of course be assured in this circle of Leibniz scholars and Leibniz fans that the subject I have selected will not appear to be out of place. In view of my ensuing remarks my references to Leibniz will not be in the nature of a cloak covering up a merely casual interest of this great man in mathematics.

There is a nice story about Hilbert. When at a gathering everyone was asked to say what question he would ask when being waked up from three hundred years' sleep of death and being permitted to ask one single question as to how things had meanwhile progressed on earth, Hilbert said he would ask whether Riemann's conjecture had meanwhile been proven. Now if Leibniz were given this opportunity here and now, he might well ask us, I think, how matters were with his *mathesis universalis*. So let's tell him!

2. Two Internal Achievements of Mathematics

To start with a formality: We already learned that from ancient time mathematics was subdivided into arithmetic and geometry. Added to them in the course of time were a few fields of application we heard Descartes mention, and in the 17th century mathematics in a narrower sense was joined by algebra and infinitesimal calculus. As far back as 1868, the yearbook on *Progress in Mathematics* subdivides mathematics (including its fields of application) into 12 subfields, followed, for greater clarity, by a still more detailed subdivision into 38 fields. In *the Mathematical Reviews* of 1979, two comparable subdivisions produce 60 and approximately 3400 subfields respectively[8]. Thus, particularly within the past 100 years, we are confronted here with an expansion and differentiation of mathematics which actually defies description: An absolutely fantastic development which even our bold prophets of a mathematical univeral science would certainly be rendered speechless. At the same time it is clear that it would be simply ridiculous to try to present, in a lecture, an adequate impression of the state of things, let alone of their development. Nevertheless, in this second part of my address, still with the whole of mathematics before our eyes; I propose the following *subdivision into three* for the consideration of us all. Unlike the classifications already mentioned, intended as means to organize the immense mass of material, our division into three is oriented to the question, just what, in a more qualitative sense, mathematics accomplishes. And here the possibility suggests itself of distinguishing between an algorithmic, a demonstrative and a descriptive accomplishment. This distinction is not one that has just become possible for modern mathematics. All three accomplishments have been known ever since antiquity, all of them are present in Leibniz's design for a universal mathematics, and each one of them has undergone a tremendous expansion since then.

Algorithms are known to us all in the form of the four fundamental rules of arithmetic with rational numbers in the decimal system. Everyone knows how two natural numbers are to be added, and if the numbers are not too large, he or she is also able to actually perform the addition. This is simply a matter of calculating the value of a function for given values of the independent variables. Another function one is taught at school to calculate is the function by which the greatest common divisor of two natural numbers is obtained: one calculates this with the aid of the so-called Euclidean algorithm. Quite generally an algorithm is a - so it is said - purely mechanical procedure which in a finite number of steps yields a well-defined result from given data. The decisive thing is that it has been prescribed by wholly unambiguous instructions just how every single step and how the sequence of steps is to be carried out. The availability of an algorithm is in the given case the compliance with Leibniz's "Calculemus!" While the pertinent basic idea is as old as elementary calculation, it is only since little more than fifty years that we have a precise *conception* of the algorithm[9]. The definition of this concept and thus the establishment of a strict science of the calculable is, in this first field of accomplishment of the mathematical, the outstanding event par excellence since the 17th century. The adequacy of the definition is expressed in Church's thesis that every intuitively calculable function is also calculable in the sense of the precise definition, a thesis which today is accepted by every mathematician.

This statement on the theory of the matter cannot be made without mentioning also the corresponding practice. It is well known that besides the, shall we say, Platonic tradition of philosophy with its high esteem of mathematics there has also been a tradition of a rather anti-mathematical orientation and that e.g. Hegel has found less than kind words on the mathematical activity of the human mind. These negative judgments pertain predominantly to the algorithmic accompl-ishment of mathematics, and in fact, of course, the mere adherence to an algorithm, once one has it, is so stupid an affair that one may assign it to a machine. On the other hand, we know better today than any preceding generation that a disavowal taking place in so isolated a fashion is totally out of place. For on the one hand the computer revolution we are witnessing today - and I believe we may really speak of a revolution here - is not, on its part, a mere algorithmic accomplishment. Rather it is a highly complicated technological development based not only on mathematical, but also on physical progress. And in any event it is based indirectly, by way of physics, on a mathematical progress which has nothing at all or little to do with algorithms. On the other hand the fact remains that the transformation of our world through the computer is based on a thoroughly effective integration of its *algorithmic capability* with other accomplishments.

I need not describe here in greater detail what undreamt-of influence modern computers are meanwhile exerting not only on our everyday life, but also on the progress of science. There is only one thing I wish to mention expressly. Normally the use of computers for scientific purposes has a *conclusive* character: within the framework of a sizable project they furnish e.g. numerical data which form a decisive part of the overall result, and this they do also e.g. in computer-assisted proofs within pure mathematics, for example in proving the Four Color Theorem[10]. In addition, however, computers also play a *heuristic* part in *research*. True, an *ars inveniendi* such as meant by Leibniz and held possible until well into the 19th century we consider today to be impossible. But that the heuristic use of computers in the recent past has brought research ahead cannot be overlooked.

A typical example is the theory of deterministic chaos[11]. Here the problem is the description of processes which obey a quite simple mathematical law, but which both in the individual case and in their totality may take place in an extremely complicated way, in a word: chaotically. To obtain an overview of such processes seems to overtax even the brains of trained mathematicians. A computer, on the other hand, gives one quite rapidly a vivid impression of the processes going on and of essential structural characteristics. Usually this is quite sufficient for the physicists, and the mathematicians will find that the theorems they will have to prove are now occurring to them. Euler is reported to have said: "If I only had the theorems already! I would have no trouble finding the proofs". At least in the first part of this task computers have an essential part today.

So much about the algorithmic accomplishment of mathematics. Now, as next thing, a word about its *demonstrative* function. Mathematics - it is said - is the proving science par excellence. What is a mathematical proof? This, too, is something most of us will probably have been confronted with at least once at school. That, of course, is not sufficient to give us an impression of the fact that the finding of proofs is the main business of mathematicians, or of how they go about it. Characteristically, however, all professional attempts undertaken so far to round up the proofs of mathematicians under a precise *concept* have failed to be as succcssful as they were in the case of the algorithm[12]. There is no Churchian theorem for the concept of 'intuitive' proof. We have several explications, but the practice of proving is not identical with any of them. In comparison, it would make little sense to apply an algorithm without, however, striving to bc absolutely precise in doing so. If we want to know the exact sum of two numbers, we must apply the rules of addition *exactly*. In contrast, mathematical proofs are often more plausibel when they do *not* exactly follow the rules of an explicit proof concept.

Nevertheless it must now be said here, too, that certain insights into the concept of proof which we have gained in the past 100 years through explicatory attempts constituted a giant step forward when these efforts are viewed in the light of Leibniz's aspirations and compared with the then state of things. The essential recognition was that to a decisive, formerly under-estimated extent the mathematical proof is simply a *logical inference*. The drawing of logically correct inferences has first of all, like calculation, the formal aspect that it occurs according to precise rules which can be combined to describe, in the aggregate, a calculatory procedure - a procedure governed by logic. Seen thus, the drawing of conclusions is, therefore, related to calculation. A big difference, however, is that the rules of calculation prescribe what - step by step - one is *obliged* to do, whereas those for drawing conclusions prescribe only what one is *permitted* to do. Permissible - roughly stated - is anything which preserves the truth - which, without limitation of the generality of truthful premises, leads to a truthful inference. The frcedom left the seer of proof within this framework, as contrasted with the blind "thou shalt" of calculation, is at the same time that which makes proving harder than calculating.

The realization that proofs are essentially logical inferences - *only* inferences - seems to reduce mathematics to applied logic, which is something mathematicians loathe to hear. In addition to that, there is the fact that the proof of a thesis, although not *being* an algorithm per se, may, in certain cases, quite well be replaced by one - by a decision procedure, as they call it here. That this trivialization of mathematics does not come to pass in the more interesting cases is expressed by a limitation theorem of Gödel[13]. The dream of a complete algorithmization of mathematics, which Leibniz, too, entertained, has been dreamt to its unsuccessful conclusion. The metamathematical analysis of proofs and possibilities of proof must, however, not be regarded anyway as an attempt to describe what mathematicians actually do. Rather, the sole issue at hand is the problem of relating mathematical proofs to a concept so that, on the basis of this proof concept, essential parts of mathematics should become reconstructible. The aforementioned solution by having recourse to logic is the best solution we know[14].

It may well come as a surprise to the outsider that the reconstruction of mathematics' proof-producing apparatus as being a logical apparatus is an insight that was gained only in the post-Leibniz period, in fact only little more than 100 years ago. Was not logic invented by as ancient a community as the Greeks, and had not, since Euclid's opus, the *demonstratio more geometrico* become a paradigm of scientific thinking? Both, the one and the other are perfectly true, and indeed logic and mathematics have continued since then to be felt time and time again to be somehow related. But this does not yet mean - far from it - that e.g. the proof given by Euclid had been expressly based on logic as it was known then. As we know today, the underdeveloped status of Greek logic at the time completely ruled out this happening in the first place. It is only toward the end of the 19th century, first and foremost in Hilbert's *Grundlagen der Geometrie* (Fundations of Geometry), that it becomes transparent that the mathematical share in geometric proofs consists of no more thgan logical conclusions from the axioms of geometry[15]. The step forward taken in this connection was a step of logic, not of mathematics. For the possibility of logical deduction is based on the occurrence, in the propositions connected by a proof; mponents of purely logical significance, such as e.g. the words 'and', 'or', 'not'. Bur for the formulation of mathematical statements and the insight into their logical interrelationships it is only the correct treatment of *generality* and *existence* - hence of the logical components of statements we express in everyday language with 'for all' and 'there is' - which is absolutely decisive. Now these statements had, however, since Aristotle, hence for more than 2000 years, been explicated only rudimentarily in the syllogistic basic forms 'B applies to all A' and 'B applies to some A'. Even the simplest theorems of geometry arc not correctly analyzable syllogistically. Unbelievable though it may sound, it was not until close to the end of the 19th century, that the mathematically fully relevant use of generality and existence was correctly recognized, particularly through the works of Frege[16], to which this and that was added later, but which undoubtely constituted the breakthrough.

3. The Description of Nature

The characterization given so far of the demonstrative power and accomplishments of mathematics is possibly incomplete. When a mathematician is asked what the purpose of a proof is it will be natural for him or her to answer that the purpose is the insight acquired in the truth of the theorem proven. He (or she) might also say that the purpose is the establishment of a logical implication: the theorem proven *follows* from these or those other theorems. This latter answer would definitely close our subject. But the former answer, the one putting the truth issue in the foreground , is heard more frequently. For those mathematicians are probably in the majority who believe that they are dealing with a mathematical subject *sui generis* and unearthing truths about it. However, a proof as described so far leads only - and this as a matter of principle - to a shift or a postponement of the truth question rather than to its resolution. For in every case the question of the truth of those propositions remains open *from* which, as premises, the proof was arrived at. If one wants more than that, the description of the demonstrative accomplishment becomes dependent of the question as to the *object* of mathematics. With this question one penetrates right into the center of the philosophic discussion of mathematics - to the question as to the - as I will call it -*descriptive power* of mathematics, which question will as of now occupy us until the end. In this third section we will first of all examine the separate, subordinated question of to what extent mathematics itself will be able to provide us with an answer.

The answer, one keeping strictly within the framework of mathematics, I will give to the question as to its subject and descriptive power, will - in accordance with this dual formulation - be a twofold one. For one thing, the descriptive power of mathematics is essentially - to put it somewhat paradoxically - an abstractive power which, in far-reaching independence of the object, presents only some such thing as its form and the form of what can be said about it, this, however, with a certain completeness in that *all* possible forms susceptible to application are indicated. In the terminology that has become customary for this accomplishment of mathematics one might express this also by saying that mathematics considers *structures* types of structures *in abstracto* . And, as we already did before in the case of algorithms and proofs, we can now also say with respect to structures and types of structures which we have developed for them in our 20th century a conceptuality granting us expanses and depths of vision which would have made the heart of a Leibniz beat faster. Similar to and in connection with the concept of proof, it again is the expansion of logic and of its languages which has made this new perspective possible. But, again, we find that here, too, the conceptuality of structure has not been definitely settled. For this, too, we have no Churchian thesis. Yet the exactness of this conceptuality will leave everything far behind it which is understood elsewhere by 'structures' - a vogue-word, a fashionable expression of the 20th century.

Above all, however, we are truly confronted here with a *mathesis universalis*: an incredibly wide formal description framework which leaves the contents to a large extent open. This framework is far wider than what Descartes understood by order and measure when he said "that, to be precise, everything must be considered as mathematics which is marked by a search for order and measure"[17]. And when he continues "that it does not at all matter here whether this measure is to be looked for in the numbers or in the figures or the stars or in the tones or in any other object", then we can, with far more right, say the same thing of the modern mathematics of abstract structures. Somewhat pithier to our understanding and illustrative of developments since then is how George Boole expressed himself 200 years later with the words:

"Anyone familiar with the present status of symbolic algebra knows that the validity of the operations of mathematical analysis does not depend on the interpretation of the symbols used (...). Every interpretation which leaves the truth of the assumed relationships intact is equally admissible, and it is in this sense that the same operation constitutes in one interpretation the solution of a problem on properties of numbers, in another one of a geometric problem, and in a third one of a problem of dynamics or optics."[18]

But Boole, too, is standing - in the mid-19th century - only at the beginning of the uninterrupted upswing toward the universal mathematics of structure. This upswing was only made possible by Cantor's theory of sets or aggregates and Hilbert's formalistic program. Under the influence of Hilbert, including his interest for the physical applications of mathematics, it thereupon was the Göttingen school of mathematicians, particularly Emmy Noether and her students, who contributed essentially to the development of the new views. Van der Waerden's *Moderne Algebra* of 1936 probably was the first textbook in the new style, with Bourbaki's mathematical encyclopaedia of the 1950s and 1960s forming the crowning conclusion[19].

Now what are structures and species of structures in the sense of modern mathematics? With a view to traditional mathematics one will assume that e.g. geometric figures - straight lines, circles, polyhedrons, etc. - are mathematical structures, as are, without a doubt, the natural numbers of arithmetic. That is quite correct, too, if in addition the following essential consideration is made: When we say of a geometric figure that it is a circle, or of a number that it is a prime number, then in doing so we are referring to a larger entity - to the system of all numbers or to space as a whole -, while furthermore applying certain universal structures to these entities - e.g. multiplication, or the function of distance - , and *without* our doing this we would be wholly unable to say anything about the individual structures, so familiar to us, of number and figure. Structures in the sense of modern mathematics are, therefore, fairly comprehensive, usually infinite formations consisting of one or more basic domains whose elements, subsets, etc. are structured by properties and relationships. Against traditional logic, the matter to be particularly emphasized here is the *many-termed (proper) relation*, which to understand was a source of difficulties until far into the 19th century. Here in the descriptive field, matters are exactly the same as they are in the logical field with respect to existence and generality: Without including *proper relations* in our considerations a reconstruction worthy of the name of scientific assertions is out of the question. The second essential insight which made the modern concept of structure possible was the inclusion into the considerations of properties and *relations of higher order*[18]. The property of being a prime number is in the system of natural numbers a property of the 1st order, since it concerns the *elements* of this system. On the other hand, the property of being a circle no longer concerns the points of the given space, but its *subsets*. Her we are dealing with a concept of the 2nd order, and even concepts of a still higher order are continually being used today in applications of mathematics. Many-termed concepts and concepts of higher order form today the germ cell for a recursive procedure for introducing within a theory of sets or a logic of types the general concept of structure[19].

Now to what extent is use being made within and outside mathematics of this newly-acquired generality? When we look first of all to the applications, the answer, in a strict sense, must be: to an infinitesimal extent. In all strictness, however, this is only intended to mean that by the very nature of things we can only make a finite use of a potentially infinite diversity of types of structures, and in principle there is nothing at all we can do to change this ratio. But in comparison with the situation in the 17th century the situation existing then has meanwhile been considerably expanded. First of all there have been expansions in the sense that wholly *new types of structures* have had to be resorted to in order to arrive at an adequate description of the objects of application. The most impressive examples of this are furnished us by physics, still constituting as it does the most mathematics-oriented empirical science we have. In generalizing the Newtonian space-time, but simultaneously in deviating from it, the general relativity theory has led us to consider the so-called Lorentzian manifolds. A particularly dramatic turn was brought about by the quantum theory, when Hildert spaces and Banach algebras were used to describe states or properties of an atom or elementary particle. This marked the first time that, to the great surprise of physicists, non-commutative algebras were introduced into physics. Likewise, the classic probability spaces resorted to to describe common statistical phenomena must be included in the list of novel structures frequently being applied today - far beyond physics - in the empirical sciences[20].

Whereas these expansions occupied physicists particularly in the first half of this century, we are since recently confronted with the realization that *internal expansions* of already known types of structures are becoming physically relevant. As an example we may mention *number-theoretical structures*. To the outsider this may sound surprising, thinking as he does that, if anything, the natural numbers have been populating physics for a long time. This is undoubtely correct, but only in the sense that, from a mathematical point-of-view, the number structures that had found application were fairly uninteresting ones. Number theory in the narrower sense has always been the l'art-pour-l'art show-object of mathematics. The British number theoretician Hardy even prided himself of the utter uselessness of his doings, and in the Anglo-Saxon realm one speaks of Hardyism as the attitude that claims the self-sufficiency of mathematics[21]. But things have changed since recently, and Steven Weinberg reported only the other day of his statisfaction over having been able, in a paper on the string theory of the elementary particles, to quote Hardy, whose determination of the so-called *partitio numerorum* - the number of additive splittings-up of a natural number - he has used in his work (23). But also into a field so close to life as room acoustics - to mention only one further example - number-theoretical structures have penetrated. To improve acoustics in modern concert halls with a too low ceiling, a ceiling profile has been proposed which follows the powers of a primitive root of a Galois field (24).

Another class of structures whose recent appearance in physics came as a surprise are the so-called *fractals* (25). If a hundred years ago mathematicians had made bets on what mathematical structures would most certainly never find application outside mathematics, highly plausible candidates for such bets would have been, for example, the so-called Cantor sets or the function, found by Weierstrass, that is continuous everywhere but nowhere differentiable. Now how do such adventurous structures ever find application? The Greeks never made even so much as a start on physics, since the natural goings-on on earth appeared immeasurable complicated to them. Newer physics lived for 300 years off the discovery that these complicated goings-on nevertheless obey simple laws. Now that we have come quite far already in knowing and understanding the laws of nature, interest is increasingly being directed toward the contingent happenings in all their complexity. And there we find e.g. in the deterministic chaos theory already mentioned that for characterizing the solution behavior of quite simple equations such exotic sets offer themselves as e.g. the aforementioned Cantorian sets (26). Such a set is arrived at by starting from a finite interval which is divided into three equal parts, of which one leaves out the middle one (without its end points), following which one performs exactly the same procedure with the two remaining intervals, then again with the intervals remaining after this second round, and so forth. The residual set will cover the original interval as thinly as desired, yet it still contains exactly as many points as the original material. The discovery of such a monstrosity was worthy of a Cantor. What we are confronted with is the problem why we find such structures in textbooks of mathematical physics.

So far I have spoken of the descriptive power of mathematics only insofar as it can be left open where the structures come from *in concreto* which mathematics considers *in abstracto*. At the close of this section a word is still needed on whether mathematics itself does not already furnish us structures. Two remarks on this question must suffice us for the following. On the one hand the remark that *models of a theory of sets* answer this question adequately at least when in the spirit of the purpose of this address present-day mathematics is regarded in the light of the idea of a *mathesis universalis*[22]. Evidently this answer is not unequivocal, but each one of its intended specifications would permit, in a superabundant measure as far as the applications are concerned, a uniform construction of mathematical structures. In this connection it is not necessary at all - and here comes the second remark - to visualize a model based on the theory of sets as a platonic heaven. Sufficient to us is the empirical fact that man is capable of the mental constructions concerned. No matter how he may have reached this point, we can furthermore note that in this spiritual world truths apply which we are able to realize without resorting to experience, without experiments and without observations on material objects and which realizations are accompanied by an uncommon measure of certainty. Now what, under these assumptions and in the light of everything said so far, does the application of mathematics to nature look like?

4. The 'Unreasonable Effectiveness' of Mathematics

Ever since the beginning of modern physics, physicists have been convinced that - as Galileo already put it - "the book of nature is written in the language of mathematics" (30). Furthermore, it has been expressed time and time again that the positive usability of mathematics for our understanding of nature borders on the miraculous. To Kepler and Galileo this miracle consisted in our being able here, if anywhere, to directly read God's thoughts. A modern physicist, Eugen Wigner, says: "The enormous usefulnss of mathematics in the natural sciences is something bordering on the mysterious, and there is no rational explanation for it" (31). The only possibility of an explanation thereupon suggested by Wigner is an aesthetic one, adopted by him from Einstein: "The observation which comes closest, to an explanation (...) is Einstein's statement that the only physical theories which we are willing to accept are the beautiful ones". But Einstein still had other things to say on the matter, and in this final section I will take up his cue and that voiced in a parallel remark by Steven Weinberg, one of the founders of the theory of electroweak interaction.

Einstein and Weinberg likewise make no secret of the fact that they find themselves confronted here with a miracle of sorts. Einstein speaks of the

"riddle which has troubled researchers of all times so much. How is it possible that mathematics, which after all is a product of human thinking independent of all experience (and whose theorems are absolutely certain and indisputable), fits the objects of the real world so perfectly?" (32).

Weinberg presents as it were an empirical confirmation of the miracle in enumerating the many cases in which a sspecies of structures used by physics had been found already before by the mathematicians and now merely needed to be correctly applied.

"It is positively spooky", says Weinberg, "how the physicist finds the mathematician has been there before him or her"[23].

The mathematician becomes so-to-speak the physicist's Man Friday. Is there any explanation for this teamplay?

Einstein has tried to solve this riddle through his now famous statement: "Insofar as the theorems of mathematics refer to reality they are not certain, and insofar as they are certain they do not refer to reality". Weinberg offers us, in contrast, the following explanation:

"Mathematics is the science of order; so perhaps the reason the mathematician discovers kinds of order which are of importannce in physics is that there are only so many kinds of order".

These two explanations seem to state wholly different things. In actual fact, however, they form part of the same picture and complement each other. Each is associated with a specific basic feature of modern universal mathematics as I pictured it: The attempted reduction of the mathematical in the proper sense to the logical-formal drawing of conclusions, thus simultaneously gaining the immense richness of possible structures which lend themselves to such drawing of conclusions. Einstein elucidates his view by remarking that it was only through modern, axiomaticly-oriented mathematics that we received absolute clarity as to the fact "that through it a clean break was achieved between the logical-formal and the objective (....) contents (and that) only the logical-formal (....) (forms) the object of mathematics". It is thus precisely through this isolation that mathematics acquires its much admired certainty. But as soon as we take mathematics out of this isolation and apply it to reality it loses this certainty, or, to put it more precisely, it acquires as applied mathematics an uncertainty: the uncertainty, namely, of the decision *which ones* of the infinitely many species of structures that *can* find application we should select in a concrete application case. This, now, is the point where Weinberg's statement intervenes. Formulated roughly, his statement says: *Some* kind of structure will do the job. It is like shopping in a department store: *Some* suit will fit. Modern mathematics offers us, in its present-day form, *all* forms of exact thinking man is capable of. By selecting one of them to use, we do the one and only thing we are in a position to do at all. And the choice we have is gigantic. Small wonder that we find the right thing.

Does the Einstein-Weinberg view explain the pre-established harmony of mathematics and reality? On this, many a thing could be said: I would like to conclude my address with the attempt to describe a difficulty which is left out in this explanation and which still surrounds the functioning of the matter with the aura of the miraculous. To begin with, it is of course correct that in comparison with the traditional stock of mathematics the immense structural richness of present-day mathematics scales down the miracle of its applicability. In the 17th century the rejection of geometry would have meant the rejection of the entire half of mathematics. One would not have known at all what to put in its place. Once, however, the new universal-mathematical perspective had been gained, the abandonment of the old geometry in favor of another one appears simply as a transition of one kind of structure to the next one. This does not mean that we or our descendants will never have to be astonished again. No one can tell whether we won't find ourselves compelled some day, for reasons coming e.g. from physics, to abandon the aforedescribed contents-oriented mathematics in favor of an alternative. In quantum field theory, and thus in a solid piece of fundamental physics, a variety of 'mathematics' is used today which does not possess a set-theoretical model, thus constituting insofar a riddle[24]. Likewise, we are acquainted today with mathematically or physically motivated expeditions into border areas of mathematics in the contemporary sense such as e.g. non-standard analysis, non-Cantorian theory of sets, multivalued logic, quantum logic and the like[25]. But on a mathematics of quantum field theory we still lack even the beginning of an idea, and the other undertakings have not, in any case, led so far to a revolution of mathematized science which one would be compelled to follow.

But also with respect to our present-day understanding of the subject there remains, as stated before, a rest. I will call it the phenomenon of *the mathematical overdetermination of physics*[26]. Roughly put, it consists in our having, in the theories of physics, frequently *more* mathematics than we can interpret physically. Let us get the genesis of this surplus straight in a very simple case, e.g. that of the state equation of a gas. With a gas equation the physicist would like to formulate a lawlike relation, valid for many gases, between pressure, volume and temperature. Although united in one gas, these quantities are rather dissimilar in nature, and at first glance it is not evident at all where a possibility should come from to formulate a relationship - any relationship - between them. The trick by which this is de facto done goes as follows: pressure, volume and temperature have this in common that their values can be described by *numbers*. Through this uniforming, that which first seemed impossible now all of a sudden becomes possible: the entire fulness of three-termed relations between numbers is available for the formulation of a gas equation. However, a price must be paid for this: these relations between numbers likewise do not gratuitously fall down from heaven; rather, they are based on the elementary calculatory operations and on the limiting processes possibly involved. And the mathematical entities thereby appearing on the scene have *no* significance in the gas theory arrived at in the given case. Hence we did not acquire our physical law here by reconstruing it as a proposition in concepts that are physically understandable throughout. Instead, we have acquired the physical structures sought for by imbedding them into richer structures at the price that *their* elements will, and even should remain physically unintelligible. And that we obtain physically useful laws in this fashion is really a miracle.

Nevertheless this miracle would not have to upset us if it were an isolated case here. In fact, however, this is only a description of what happens *normally*. It is wholly normal that in physical theories - semantically formulated - terms occur for which no physical significance, however indirect a significance may be, has ever been even so much as intended, although these terms occur in a descriptive position. Anyone not knowing how the formalism is to be interpreted in the first place might well regard these de facto non-interpreted terms with equal justification as interpreted as the actually interpreted ones. For this reason there can, at first glance and without further consideration, be no question of the borderline between form and contents coinciding, according to Einstein's ideas, with that between mathematics and physical reality. Rather, theories formulated in this fashion are mixed forms which describe a material world by relating it to a mathematical one.

Do we now also have an explanation for the phenomenon of mathematical overdetermination? It is noteworthy that the attempts at an explanation have mainly consisted in causing the phenomenon to disappear, i.e. in showing that theories manifesting it possess physically equivalent formulations from which it is eliminated[27]. Paradigmatic for this continues to be, even to this day, Euclidean geometry. Its modern version, preferred in physics, as analytic geometry employs coordinate systems in space and thus numbers. It can be shown, however, that one can also do without this analytical apparatus and that an equally strong formulation in purely geometric concepts exists[28]. We will consider another case somewhat more precisely. Geometry is, in the common view, equipped with a distance concept which lets the distance between any two points in space be an unequivocally determined number. This distance structure contains somewhat *more* than is given in a physically objective fashion. We will obtain a specific number only if we *arbitrarily* lay down a unit of measure. Objectively given is only the equality of two distances: the socalled congruency. Now it is indeed posssible to present a formulation of Euclidean geometry which proceeds exclusively from the congruency and betweenness relations and from which distance numbers have disappeared. What has thereby been achieved? When we say that the distance from Hannover to Heidelberg measures some 400 km we have interrelated two places on our planet by a number. It is difficult to argue the fact out of existence that into this distance relationship the number concerned enters in exactly the same fashion as the two spatial partners. Now two places materially defined in space are just as certainly physical realities as a number - the third partner in our relationship - is *not*. Why is it necessary to talk in physics, besides on material realities (in a broad sense), also something entirely different, e.g. on numbers? One is tempted to answer that there is something wrong here already in the very question - that the numbers do in fact play a different role in the given theory than its actual objects. That may well be so. But unfortunately we do not possess a reconstruction which would make this difference plain and *thus* explain our phenomenon. In the given case we can instead make the phenomenon disappear: as stated before, things will work *here* also without distance numbers. But is this answer satisfactory and will this always work?

Both questions, I am afraid, must be denied. The newer field theories, including the quantum field theories, have all been formulated with space-time coordinate systems being resorted to. Now even many physicists have a tendency to keep the further development of these theories free of coordinates. But this does not remove the sting placed here in the very beginning. From the part of philosophy of science, the attempt was recently made to eliminate, by the same process as just outlined for the distance function, numerical values also from true field functions (37). The result is in these cases of appalling complexity. A preferred object of reaxiomatizing attempts has been furthermore, ever since ist physical establishment 60 years ago, quantum mechanics. Its original formulation, used today in all textbooks, possesses a not even particularly conspicuous, but - in its consequences - far-reaching mathematical overdetermination in the form of complex Hilbert space. Here the reformulations have frequently been attempted for wholly different purposes and, accordingly, have yielded nothing that would help us in our question. Other attempts have not yet been sufficiently clarified to permit a clear decision as to their success[29]. From the point of view of physics as a whole, all these undertakings are only punctual in nature, even though the points where they are undertaken may be crucial ones. If nevertheless we wish to draw a lesson from them already now, we seem to find the rule confirmed that the attempted economizing on ontological assumptions, hence here the avoidance of mathematical entitites in the position of objects - if practicable at all - frequently leads to undesirable complications. But this rule, too, cannot yet be considered as fully understood.

Where - so I ask in definite conclusion - has this investigation led us? I have tried to outline in what the decisive advancements of logic, mathematics and their applications since Leibniz's times can be seen to lie if developments are viewed in the light of the idea of a universal language and universal science. For this purpose, three domains of accomplishment or power were distinguished.

The *algorithmic* success is the most conspicuous one: Leibniz's little calculation machine has been replaced by our worldwide, even satellite-wide integrated large-scale computing systems. And these systems can calculate anything regarded as theoretically calculable today.

The success achieved in the field of *proof theory* consists above all in logic having caught up with mathematics, so that appreciable parts of the latter can now be treated axiomatically. This did not involve, however, a complete reduction of mathematics to logic.

Noteworthy, finally is the immense gain in *descriptive potency* and the updating thereof, which appear to express the universality of present-day mathematics most clearly. Quite a few things have come to pass here which no one could foresee in the 17th century.

Other hoped-for things have not been realized. In all, mathematics has achieved greater independence vis-à-vis other forms of knowledge, thus netting us the so-called application problem. This wide-branching problem I have pursued only along o n e line. Starting out from the amazement at the "unreasonable effectiveness of mathematics", as Wigner calls it, I have described an attempt at a solution which starts out from the universalistic gains achieved by modern mathematics. We found, however, that difficulties are encountered here, which to overcome has, admittedly, been attempted but not yet really achieved. The difficulty here is that mathematics is more than logic and shows us its teeth on the descriptive level. Thus an important idea, which Leibniz, as an early forerunner of logicism, had entertained, too, has not been fulfilled. That, too, we would have therefore have to tell him in our story. If he were not merely able to ask us the one question we started out by permitting him to ask, but also capable of counseling us in this situation, we would not be assembled here and now in so large a number without lending him our ears.

Notes

* Translation of the paper *Calculemus! Das Problem der Anwendung von Logik und Mathematik* given at the Leibniz Congress 1988 and printed in Studia Leibnitiana, Suppl. XXVII. Stuttgart: Franz Steiner Verlag 1990. p.201-216. We gratefully acknowledge permission to translate and print the article by both, author and publisher.

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1 R.Descartes, Oeuvres, ed. by Ch.Adam and P.Tannery, Paris 1897-New Ed. ibid. 1974-. Here: vol.I, p.80-. Cf. also G.W.Leibniz, Vorausedition der Philosophischen Schriften, Fasc.7, Münster 1988, p.1480-.

2 G.W.Leibniz, Opuscules et fragments inédits, ed. by L. Couturat, Paris 1903, Hildesheim 1961. Here: p.155

3 Ibid. p.156. See also ibid. p.176 and C.J.Gerhardt, Die philosophischen Schriften von G.W.Leibniz, Berlin 1890, Hildesheim 1961. Here: p.124-, 198-. I owe the reference to the collection of Calculemus citations to Hidé Ishiguro.

4 Leibniz, op.cit. no.2, p.284-.

5 G.W.Lllcibniz, Sämtliche Schriften und Briefe (Akademie-Ausgabe), ser.I, vol.11, Berlin 1982. Here: 420-

6 For an interpreation of the relevant undertakings of Leibniz and his contemporaries see (1) and compare also (2).

7 Descartes, op.cit. no.1, vol.X, p.377

8 See (3)

9 See (4) and (5) where the basic works have been printed. See also (6)

10 See (7)

11 See for instance (8)

12 As an introduction into the "many faces" of logic (9) is useful and for further reading take e.g. (10).

13 See the references of note no.9.

14 On the concept of rational reconstruction in comparison to history of science see (11)

15 See D.Hilbert in (12) and compare the description of the development in (13).

16 See Frege in (14) and (15)

17 Descartes, op.cit. no.1, vol.X, p.378

18 We owe Frege a modern understanding of relations and the introduction of concepts of a higher order, see Note 16 as well as later on Whitehead and Russell in (19).

19 For a set-theoretical introduction see Bourbaki (20), for a modeltheoretical introduction see (10) vol.I, Ch.4.

20 Textbooks of theoretical physics consider the modern understanding of mathematics too, see e.g. (21).

21 See Hardy (22) as well as (3).

22 See e.g. Jensen (29)
23 Compare (23)p.725-, sec also (33)
24 Physicists (rightfully) disregard this circumstance, as they are verysuccessful with their method in the (so-called)renormalization theories.
25 Compare here the references under note 12.
26 Concerning the following see (34).
27 This reaxiomatization was firstformulated as a program in the beginning of reference (35).
28 On this and on the following case see e.g. (36)
29 In this regard I am thinking above all of Ludwig (38).

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