Rudolf Wille
Technical University, Darmstadt, FRG

Line Diagrams of Hierarchical Concept Systems

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The possibility of visualizing conceptual relationships by graphic representation of hierarchies has been used in the standard DIN 2331 and is demonstrated here using a set-theoretical model for hierarchical concept systems. For a fixed context it is provable that the concepts in generic relationship (subconcept - superconcept) form – as a mathematical structure – a complete lattice called the "concept lattice" of the context. Methods and results of order and lattice theory can thereby be used for concept analysis. Line diagrams, as described in DIN 2331, are studied and used a great deal in lattice theory. By introducing graded line diagrams, the use of this representation method can be considerably extended, to the point where it is possible to visualize hierarchies with several hundred concepts. It is also shown that building blocks of modified line diagrams may be understood as scales in the sense of a conceptual measurement theory. (Author)

0. Introduction

The German standards DIN 2330 "Begriffe und Benennungen; Allgemeine Grundsätze" (4) and DIN 2331 „Begriffssysteme und ihre Darstellung" (5) were written to aid the use and understanding of conceptual tools in sciences, industry, and administration. The definitions of a concept and a concept system, which the standards have taken over from traditional philosophy, may be formulated within a set-theoretical model with respect to a fixed context. It is then provable that the concepts of a context together with the hierarchical relation "subconcept-superconcept" form as a mathematical structure a complete lattice called the "concept lattice" of the context. Methods and results of order and lattice theory[1] can thereby be used for concept analysis. This idea shall be discussed in this note with respect to the graphic representation of concept systems. The line diagrams described in DIN 2331 are studied and used a good deal in lattice theory (12). By introducing graded line diagrams, the use of this method of representation can be considerably extended, to the point where it is possible to visualize hierarchies with several hundred concepts. The graded line diagram is based on the construction of subdirect products. There are also other construction methods in lattice theory which yield useful modifications of line diagrams, a fact which shall be demonstrated by the sum of an atlas of lattices. The building blocks of a modified line diagram may be understood as scales in the sense of a conceptual measurement theory. In closing, this final aspect of modified line diagrams shall be discussed.

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1. Concept Lattices

In concept theory, a concept is viewed as a pair consisting of an extension and an intension [comprehension]: the extension consists of all objects (or entities) belonging to the concept, while the intension comprises all attributes (or properties) valid for all those objects (16). In handling concepts and relations between concepts, the difficulty often arises that the extension and the intension of a concept cannot be completely described (consider the concept "human being", for example). For this reason, DIN 2330 proposes to restrict to those attributes which are clearly identified with the objects under consideration. The set-theoretical model for hierarchical concept systems is based on the following definition (18) which incorporates this limitation: A *context* is a triple (G, M, I) where G and M are sets, and I is a binary relation between G and M (i.e. $I \subseteq G \times M$); the elements of G and M are called *objects* and *attributes*, respectively. If gIm holds (i.e. $(g,m) \in I$), we say: the object g *has* the attribute m. The following abbreviations have proved to be useful: For $A \subseteq G$ and $B \subseteq M$ we define

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\},$$
$$B' := \{g \in G \mid gIm \text{ for all } m \in B\}.$$

The mappings given by $A \mapsto A'$ and $B \mapsto B'$ form a Galois connection between the power sets of G and M which is characterized by the following properties:

(1) $A_1 \subseteq A_2$ implies $A_1' \supseteq A_2'$ for $A_1, A_2 \subseteq G$,

(1′) $B_1 \subseteq B_2$ implies $B_1' \supseteq B_2'$ for $B_1, B_2 \subseteq M$,

(2) $A \subseteq A''$ and $A' = A'''$ for $A \subseteq G$,

(2′) $B \subseteq B''$ and $B' = B'''$ for $B \subseteq M$.

Now, for the set-theoretical model, the following definition of a concept suggests itself: A *concept* of a context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$; A and B are called the *extent* and the *intent* of the concept (A, B), respectively. The set of all concepts of (G, M, I) is denoted by $\mathfrak{B}(G, M, I)$. A concept (A_1, B_1) is said to be a *subconcept* of the concept (A_2, B_2) (in symbols: $(A_1, B_1) \leqslant (A_2, B_2)$) if $A_1 \subseteq A_2$, which is equivalent to $B_1 \supseteq B_2$ by (1) and (1′); in this case (A_2, B_2) is called a *superconcept* of (A_1, B_1). The mathematical structure $\underline{\mathfrak{B}}(G, M, I) := (\mathfrak{B}(G, M, I), \leqslant)$ can be viewed as a model for a hierarchical concept system.

Basic Theorem for Concept Lattices (18): Let (G, M, I) be a context. Then $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice, called the *concept lattice* of (G, M, I), for which the infimum and the supremum can be described as follows:

$$\bigwedge_{t \in T} (A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcap_{t \in T} A_t)'),$$

$$\bigvee_{t \in T} (A_t, B_t) = ((\bigcap_{t \in T} B_t)', \bigcap_{t \in T} B_t).$$

In general, a complete lattice L is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there exist mappings $\gamma : G \to L$ and $\mu : M \to L$ such that γG is *supremum-dense* in L (i.e., L =

$\{\bigvee X \mid X \subseteq G\})$, μM is *infimum-dense* in L (i.e., $L = \{\bigwedge X \mid X \subseteq \mu M\})$, and g I m is equivalent to $\gamma g \leqslant \mu m$; in particular, $L \cong \mathfrak{B}(L, L, \leqslant)$.

One of the basic tasks is to determine and to visualize the concept lattice $\mathfrak{B}(G, M, I)$ for a given context (G, M, I). By (2) and (2′), all concepts of (G, M, I) can be derived by forming the pairs (X'', X') for $X \subseteq G$ and the pairs (Y', Y'') for $Y \subseteq M$, respectively. The expense to consider all subsets of G and M may be reduced by applying the following formulas, respectively:

$$X' = \bigcap_{g \in X} \{g\}', \qquad Y' = \bigcap_{m \in Y} \{m\}'.$$

Useful algorithms for determining all concepts of a context are available in the form of computer programs[2]. The visualization of concept lattices by *line diagrams*, which are also called *Hasse diagrams* or *order diagrams* in mathematics, shall be demonstrated by an example. The cross table in Fig.1, which underlies the conceptual structure of a Hungarian educational film "Living Beings and Water" (1) (15), describes a context (G, M, I) with G:= {leech, bream, frog, dog, spike-weed, reed, bean, maize} and M:= {needs water to live, lives in water, lives on land, needs chlorophyll to prepare food, two little leaves grow on germinating, one little leaf grows on germinating, can move about, has limbs, suckels its offsprings} ; the crosses describe the relation I. In the line diagram of Fig. 2, the small circles stand for the concepts of the context (G, M, I); their extents and intents are specified by numbers and by letters attached to the circles, respectively. A concept is a (proper) subconcept of another concept if and only if an ascending path of line segments leads from the circle of the first concept to the circle of the second concept. The line diagram becomes clearer when the label of an object g and of an attribute m is attached only to the circle of the concept $(\{g\}'', \{g\}')$ and $(\{m\}', \{m\}'')$, respectively, as in Fig. 3. Note that it is still possible to read the context, the extents, and the intents from the line diagram because the mappings $\gamma : G \to L$ and $\mu : M \to L$ specified by the Basic Theorem for Concept Lattices are definable in the case $L := \mathfrak{B}(G, M, I)$ by $\gamma g := (\{g\}'', \{g\}')$ for $g \in G$ and by $\mu m := (\{m\}', \{m\}'')$ for $m \in M$. For instance, the small circle in the center of Fig. 3 represents the concept with the extent {3, 6} and the intent {a, b, c} since paths of line segments descend only from this circle to 3 and 6 and ascend only to a, b, and c.

2. Graded Line Diagrams

The clarity of line diagrams diminishes as the number of conceptual relations increases; too many crossings of line segments make it very difficult to identify single line segments. Consequently graded line diagrams will be introduced to allow a readable visualization of concept lattices with several hundred concepts. The basic idea of graded line diagrams is to delimit parts of the line diagram and to replace (parallel) line segments between these parts by a single line segment. First the graded line diagram shall be illustrated by a small example: the concept lattice of the context in Fig. 4 (9). The graded line diagram in Fig. 6 consists of four delimited *panes*. A *connecting line segment* between two congruent panes indicates that two circles are to be joined by a line

	a needs water to live	b lives in water	c lives on land	d needs chlorophyll to prepare food	e two little leaves grow on germinating	f one little leave grows on germinating	g can move about	h has limbs	i suckels its offsprings
1	x	x					x		
2	x	x					x	x	
3	x	x	x				x	x	
4	x		x				x	x	x
5	x	x		x		x			
6	x	x	x	x		x			
7	x		x	x	x				
8	x		x	x		x			

Fig. 1: Context of an educational film "Living Beings and Water"

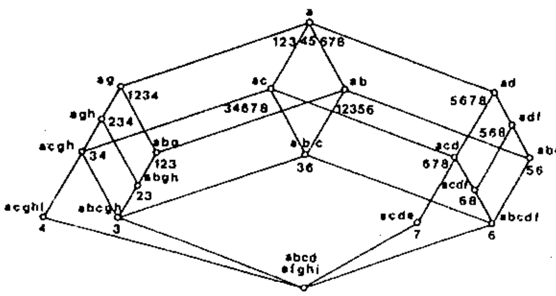

Fig. 2: Concept lattice of the context in Fig. 1

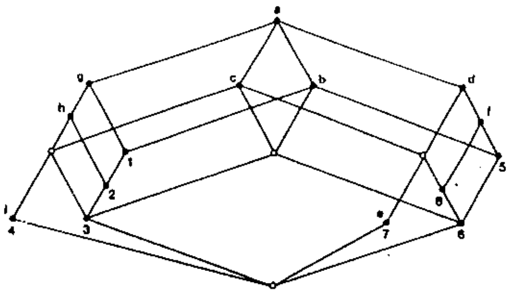

Fig. 3: Concept lattice of Fig. 2 with reduced labelings

	floating	stagnant	natural	artificial	large	small
river	x		x		x	
brook	x		x			x
canal	x			x	x	
ditch	x			x		x
lake		x	x		x	
pond		x	x			x
basin		x		x	x	
pool		x		x		x

Fig. 4: Context for the lexical field "waters"

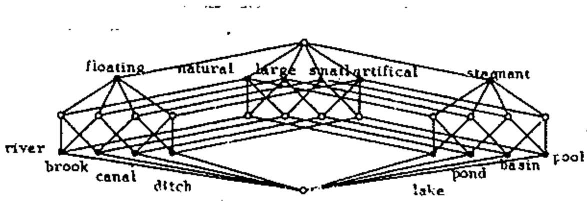

Fig. 5: Line diagram of the concept lattice of the context in Fig. 4

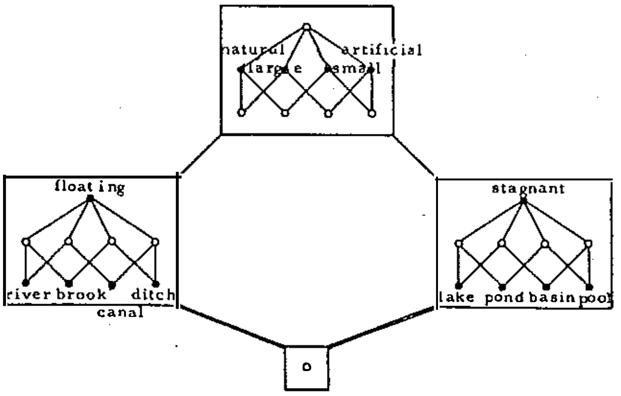

Fig. 6: Graded line diagram of the concept lattice of the context in Fig. 4

segment if they coincide after translating one pane onto the other; in this way one obtains the usual line diagram. A *double line segment* between two panes indicates that a line segment is to be added from each circle in the lower pane to each circle in the upper pane[3]. It is not necessary that congruent panes contain the same line diagrams; this fact is shown by an example taken from medicine[4] (Fig.7). If one translates all congruent panes of the graded line diagram in Fig. 8 onto one of the panes, the individual line diagrams join together to form the line diagram of the lattice of all subsets of a four-element set. The reason for this lies in the fact that the subcontext in Fig. 7 with {e, f, g, h } as a set of attributes has a concept lattice isomorphic to the lattice of all subsets of {e, f, g, h}. The subcontext with {a, b, c, d} as set of attributes has a concept lattice which is visualized by the ten panes and their connecting line segments in Fig. 8. Thus the graded line diagram arises from a *partition of the set of all attributes*: One subset yields the scheme for the individual line diagrams in the panes, and the other subset gives the diagram with the panes as elements. Different partitions of the set of all attributes usually lead to different graded line diagrams which often afford new insights into the concept lattice.

The drawing of a graded line diagram may be profitably connected with the determination of the concept lattice. If the set of all attributes of a context (G,M,I) is partitioned into subsets M_1 and M_2, one may determine first the concept lattices $\mathfrak{B}(G, M_1, I \cap G\times M_1)$ and $\mathfrak{B}(G, M_2, I \cap G\times M_2)$. Then $\mathfrak{B}(G, M_2, I \cap G\times M_2)$ may be drawn as a diagram of panes in each of which the line diagram of $\mathfrak{B}(G, M_1, I \cap G\times M_1)$ is fixed in a congruent manner (eventually without the least element). Next the small circle for each concept γg ($g \in G$) can be located in the pane representing $((\{g\}' \cap M_2)', \{g\}' \cap M_2)$ as the circle corresponding to $((\{g\}' \cap M_1)', \{g\}' \cap M_1)$. For instance, if g is the object 33 in Fig. 7, one may find the small circle for γg in Fig. 8 in the upper left pane because $\{g\}' \cap M_2 = \{a\}$ and within this pane on the extreme right because $\{g\}' \cap M_1 = \{e, h\}$. Since γG is supremum-dense in $\mathfrak{B}(G,M,I)$ by the Basic Theorem for Concept Lattices, all circles of the graded line diagram are obtained by forming all suprema of the already located concepts γg ($g \in G$).

The question remains: How can one suitably partition the set of all attributes of a context? The answer naturally depends on what one wishes to emphasize in the conceptual structure of the context. In an example from archeology (Fig. 9) (6) (10), this question shall be discussed only from the aspect of how one may obtain a clear drawing of a large concept lattice. The basic idea is to provide the set of all attributes with as many structural ingredients as are needed to find a suitable partition. Most important is the *order relation* defined by $m_1 \leq m_2 :\Leftrightarrow \{m_1\}' \supseteq \{m_2\}'$ ($\Leftrightarrow \mu m_1 \leq \mu m_2$). The ordered set (M,≤) for the context in Fig. 9 is drawn in Fig.10. The partition of the attributes into {g,h}, {i,j,k,l}, and {a,b,c,d,e,f,m,n,o,p} yields the 3-graded line diagram in Fig. 11. The two chains {c,e,o,m,a} and {b,n,p,f,d} have been purposely combined into one class of the partition since the subcontext determined by this class has a concept lattice with a clear drawing. In general, it is best to partition the set of attributes into as few chains as possible and to form the classes of the desired partition out of combinations of these chains. In searching for clear diagrams, knowing the closures of pairs of attributes might also be helpful; they are given as the outcome of the operation $\{m_1, m_2\} \mapsto \{m_1, m_2\}''$. For the context in Fig. 9 this operation is listed in Fig. 12 (trivial closures are omitted). As one sees from the table in Fig. 12, the class {i,j,k,l} combines two pairs of attributes having the same closure and thus yielding a simple line diagram. The other classes of the given partition also consist of pairs of attributes with the same closure.

closure	generating pairs
M	ab, bc, be, bh, bi, bm, bo, cd, cf, ci, cn, cp, ef, ei, en, ep, gh, ij, kl, mn, no, op
acehikmo	ch, eh
adfh	dh, fh
adfhl	al, hl
adfhp	ap, hp
adfhpn	an, hn
agkmo	ag, gm, go
ajmo	jm, jo
akmo	km, ko
bdfgjlnp	bl, gl
dfjnp	jn, jp
dfknp	kn, kp

Fig. 12: Closure operation applied to peirs of attributes of the context in Fig. 9

It is planned to automate the drawing of graded line diagrams following the sketched method, for which a set of standard diagrams shall form a starting basis. Such diagrams may be viewed as graphic scales of measurement, an idea outlined further in section 4.

3. Line Diagrams Reflecting Lattice Constructions

A graded line diagram of a concept lattice $\mathfrak{B}(G,M,I)$ in its described form is an expression of a subdirect product representation of $\mathfrak{B}(G,M,I)$ as ∨-semilattice. This is the content of the following theorem:

Theorem (18): Let $\{M_1, M_2, ..., M_n\}$ be a partition of the set of all attributes of the context (G,M,I). Then

$$(A,B) \mapsto (((B \cap M_1)', B \cap M_1), ..., ((B \cap M_n)', B \cap M_n))$$

describes an isomorphism from $\mathfrak{B}(G,M,I)$ onto a subdirect product of the ∨-semilattices $(G, M_i, I \cap G\times M_i)$ $(i=1, ..., n)$.

Besides the subdirect product construction, the sum of an atlas of complete lattices also yields a useful

	number	a	b	c	d	e	f	g	h
1	1212	x							
2	227								x
3	261		x						
4	45				x				
5	29					x			
6	136			x					
7	83						x		
8	138							x	
9	11		x					x	
10	11				x			x	
11	123	x				x			
12	220	x							x
13	15		x			x			
14	27			x		x			
15	64			x					x
16	9					x			x
17	23				x				x
18	104	x		x					
19	18	x			x				
20	15			x			x		
21	36		x						x
22	39	x					x		
23	8		x	x					
24	12						x		x
25	117	x						x	
26	7		x				x		
27	16							x	x
28	3					x	x		
29	4						x	x	
30	1				x		x		
31	2		x		x				
32	1					x		x	
33	20	x				x			x
34	31	x		x		x			
35	1			x			x	x	
36	5		x	x		x			
37	9			x			x		x
38	2			x		x		x	
39	1		x			x		x	
40	13			x		x			x
41	3		x					x	x
42	2		x				x		x
43	36	x		x					x
44	17	x					x		x
45	9		x	x					x
46	27	x						x	x
47	1		x	x			x		
48	4			x				x	x
49	2		x			x			x
50	1		x	x				x	
51	1						x	x	x
52	6	x					x	x	
53	6	x				x		x	
54	2	x		x				x	
55	15	x			x				x
56	6	x		x			x		
57	3	x				x	x		
58	1				x	x			x
59	1					x		x	x
60	1		x			x	x		
61	1			x		x	x		
62	1	x			x		x		
63	2		x		x				x
64	1					x	x		x
65	2				x			x	x
66	1	x			x			x	
67	5			x		x		x	x
68	1			x		x	x	x	
69	3	x					x	x	x
70	5		x	x		x			x
71	4	x		x		x	x		
72	9			x		x	x		x
73	1		x			x	x		x
74	1	x			x			x	x
75	12	x		x		x			x
76	2	x		x				x	x
77	1			x			x	x	x
78	2	x		x		x		x	
79	1	x		x			x		x
80	1	x		x			x	x	
81	1	x				x	x		x
82	1	x				x	x	x	
83	3	x		x			x	x	x
84	2	x		x		x	x		x
85	2	x		x		x		x	x
86	2	x		x		x	x	x	
87	4			x		x	x	x	x
88	2	x		x		x	x	x	x

a: emotionally handicapped
b: mentally ill
c: handicapped by brain injury
d: disabled
e: epileptic
f: visually handicapped
g: deaf
h: suffering from a somatic heart disease

Fig. 7: Context of some research on handicapped children

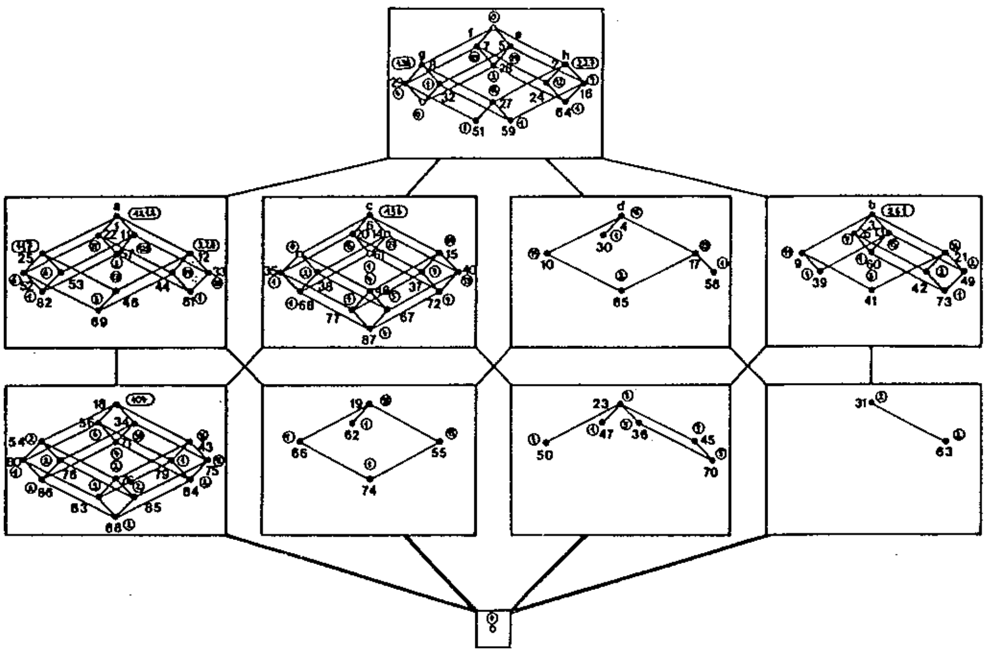

Fig. 8: Concept lattice of the context in Fig. 7

	position of handle		surface of handle		fastening of handle		form of knob on cover		decoration on cover		neck on cover		ears on cover		profile of vessel	
	lateral	trans-versal	twisted	smooth	with rings	direct	round	flat	absent	present	present	absent	absent	present	normal	flattened
	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
1	x		x		x		x		x		x		x		x	
2	x		x		x			x	x		x		x		x	
3	x			x	x		x		x		x		x		x	
4	x			x		x	x		x		x		x		x	
5	x		x		x		x			x	x		x		x	
6	x			x	x		x			x	x		x		x	
7	x			x		x	x			x	x		x		x	
8	x			x		x		x		x	x		x		x	
9	x			x		x		x		x		x	x		x	
10	x			x		x		x	x		x		x		x	
11	x			x		x		x	x			x	x		x	
12	x			x		x		x	x			x	x			x
13	x			x		x		x	x		x			x		x
14	x			x		x		x	x			x		x		x
15	x			x		x		x		x		x		x		x
16		x		x		x	x			x	x			x		x
17		x		x		x	x			x		x		x		x

Fig.9: Context of Chinese urns (14th – 9th century B.C.)

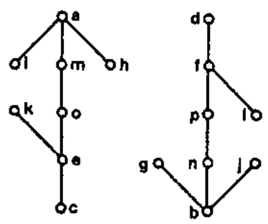

Fig.10: The ordered set of all attributes of the context in Fig. 9

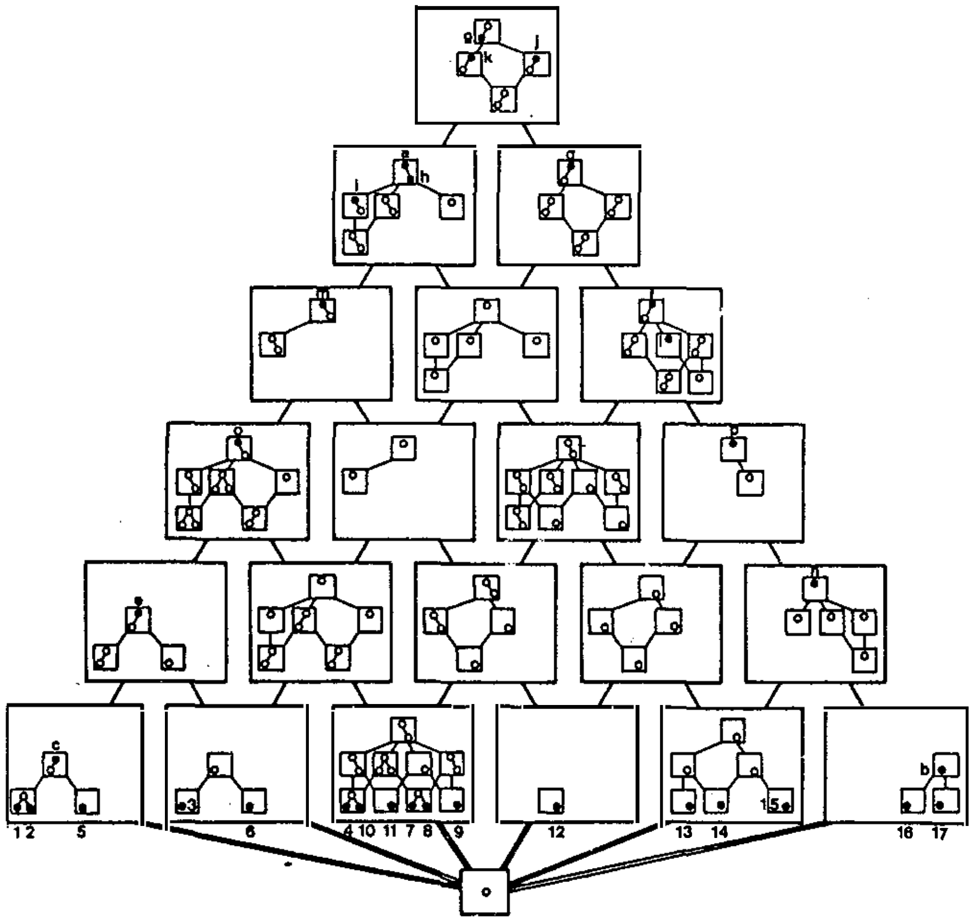

Fig.11: Concept lattice of the context in Fig. 9

	A	B	D	C	A ∘ (B ∩ C)	B ∘ (A ∩ C)	C ∘ (A ∩ B)	A ∘ (B ∩ D)	B ∘ (A ∩ D)	D ∘ (A ∩ B)	A ∘ B	A ∘ C	B ∘ C	A ∘ (D ∩ B ∘ C)	B ∘ (D ∩ A ∘ C)	D ∘ (A ∩ B ∘ C)	A ∘ B ∘ C	A ∘ D	B ∘ D
A	x				x			x			x	x	•	x	•	•	x	x	•
B		x				x			x		x	•	x	•	x	•	x	•	x
C			x	x			x			x		x	x	x	x	x	x	x	x
D			x	•			•			x		•	•	•	•	x	•	x	x
A ∩ B ∘ D	x				x			x			x	x	•	x	•	•	x	x	x
B ∩ A ∘ D		x				x			x		x	•	x	•	x	•	x	x	x
D ∩ A ∘ B			x	•			•			x	x	•	•	•	•	x	x	x	x
A ∩ B ∘ C	x				x			x			x	x	x	x	x	x	x	x	x
B ∩ A ∘ C		x				x			x		x	x	x	x	x	x	x	x	x
C ∩ A ∘ B			x	x			x			x	x	x	x	x	x	x	x	x	x
A ∩ B	x	x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
A ∩ D	x	•	x	•	x	•	•	x	x	x	x	x	•	x	x	x	x	x	x
B ∩ D	•	x	x	•	•	x	•	x	x	x	x	•	x	x	x	x	x	x	x
A ∩ C ∘ (B ∩ D)	x	•	x	•	x	•	•	x	x	x	x	x	x	x	x	x	x	x	x
B ∩ C ∘ (A ∩ D)	•	x	x	•	•	x	•	x	x	x	x	x	x	x	x	x	x	x	x
C ∩ A ∘ (B ∩ D)	•	•	x	x	•	•	x	x	x	x	x	x	x	x	x	x	x	x	x
A ∩ B ∩ D	x	x	x	•	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
A ∩ C	x	•	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
B ∩ C	•	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

Fig. 13: Context of a 4-generated experimental design (With a block relation)

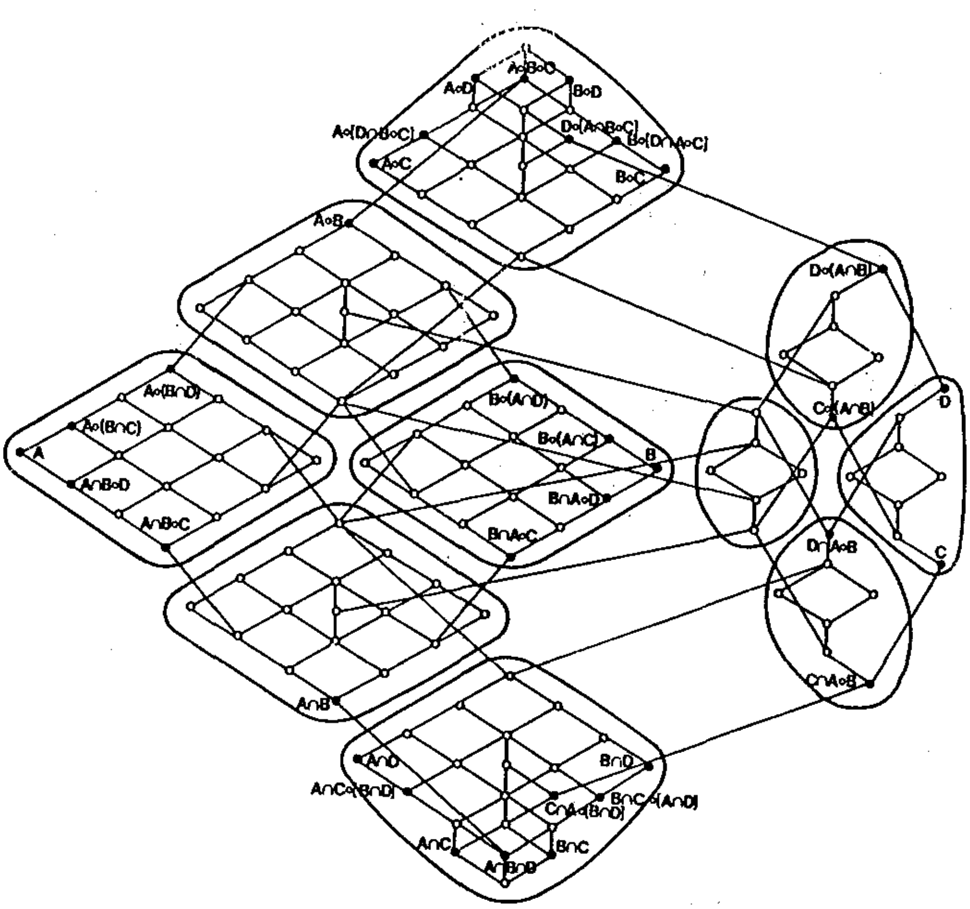

Fig. 14: Divided line diagram of the concept lattice of the context in Fig. 13

	harmonic forms	∅	1	12	13	14	123	124	125	126	135	$\overline{135}$	$\overline{126}$	$\overline{125}$	$\overline{124}$	$\overline{123}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	$\overline{1}$	$\overline{\emptyset}$
∅	rest	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
1	unison	•	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
12	second		•	x	•	•	x	x	x	x	•	x	x	x	x	x	x	x	x	x	x
13	third		•	•	x	•	x	x	•	x	x	x	x	x	x	x	x	x	x	x	x
14	fourth		•	•	•	x	•	x	x	x	x	x	x	x	x	x	x	x	x	x	x
123	9th chord without 5th and 7th						x	•		•		•	x	•	x	x	x	x	x	x	x
124	7th chord without 5th							x		•		x	x	x	•	x	x	x	x	x	x
125	9th chord without 3rd and 7th							•	x	•		•	x	x	x	•	x	x	x	x	x
126	7th chord without 3rd							•		x		x	•	x	x	x	x	x	x	x	x
135	triad							•		•	x	x	x	•	x	•	x	x	x	x	x
$\overline{135}$	compact seventh chord											x					x	x	•	x	x
$\overline{126}$	9th chord without 7th							•		•		•	x	•	•	•	x	x	x	x	x
$\overline{125}$	11th chord without 5th and 9th													x			x	•	x	x	x
$\overline{124}$	9th chord without 3rd							•		•		•	•	•	x	•	x	x	x	x	x
$\overline{123}$	9th chord without 5th															x	•	x	x	x	x
$\overline{14}$	11th chord without 9th																x	•	•	x	x
$\overline{13}$	compact 9th chord																•	x	•	x	x
$\overline{12}$	11th chord without 7th																•	•	x	x	x
$\overline{1}$	compact 11th chord																•	•	•	x	x
$\overline{\emptyset}$	compact 13th chord																			•	x

Fig.15: Context of diatonic harmonic forms (with a block relation)

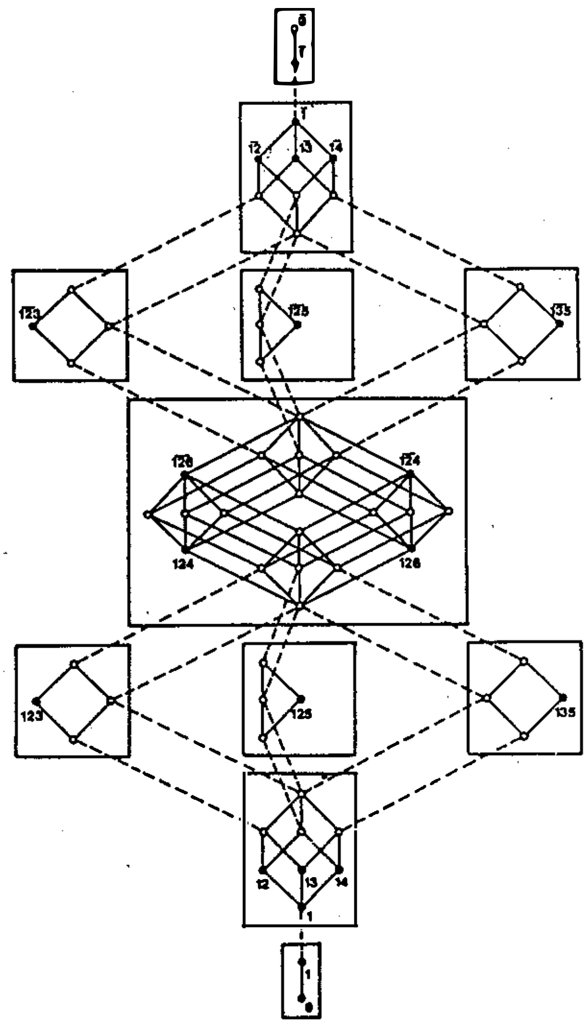

Fig.16: Divided line diagram of the concept lattice of the context in Fig.15

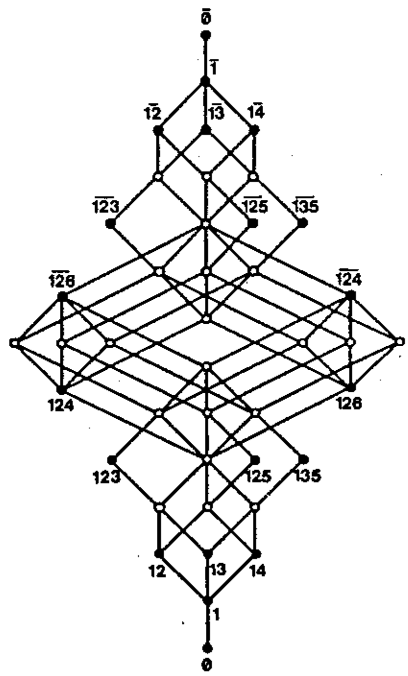

Fig.17: Line diagram of the concept lattice of the context in Fig. 15

modification of the line diagram. The name of this construction already indicates a pictorial idea. Indeed, the analogy to a road atlas helps in understanding this construction method. Analogous to the general map, there is a complete lattice Q which is the basis for the sum. For each $q \in Q$ there is as a "map" a complete lattice. L_q. The connection between two "maps" L_q and L_r with $q \leqslant r$ is established by a $\bigvee$-morphism φ^r_q: $L_q \to L_r$ and a $\bigwedge$-morphism ψ^r_q: $L_r \to L_q$. The family $(L_q, \varphi^r_q, \psi^r_q)$ is called a *Q-atlas* if suitable assumptions guarantee that $(\bigcup_{q\in Q} L_q, \sqcup, \sqcap)$ is a complete lattice with $\bigsqcup_{t\in T} x_t := \bigvee_{t\in T} \varphi^r_{q_t} x_t$ as suprema and $\bigsqcap_{t\in T} x_t := \bigwedge_{t\in T} \psi^s_{q_t} x_t$ as infirma where $x_t \in L_{q_t}$ $(t\in T)$, $r := \bigvee_{t\in T} q_t$, and $s := \bigwedge_{t\in T} q_t$ (21). The complete lattices L_q $(q \in Q)$ are named the *maps* of the Q-atlas and the complete lattice $(\bigcup_{q\in Q} L_q, \sqcup, \sqcap)$ is called the *sum* of the Q-atlas.

In order to explain the modified line diagram for the sum of a Q-atlas with respect to concept lattices, we first recall how the possible sum decompositions of a concept lattice $\mathfrak{B}(G,M,I)$ can be seen within the context (G,M,I). These decompositions correspond to the so-called block relations J of (G,M,I) which are characterized by the following properties: $I \subseteq J \subseteq G\text{x}M$, for all $m \in M$, $\{g \in G \mid gJm\}$ is an extent of (G, M, I) and, for all $g \in G$, $\{m \in M \mid gJm\}$ is an intent of (G, M, I). The manner of which a block relation gives rise to a sum decomposition is the content of the following theorem:

Theorem (21): Let J be a block relation of the context (G,M,I). Then $\mathfrak{B}(G,M,I)$ is the sum of the Q-atlas with $Q := \mathfrak{B}(G,M,I)$, $L_q := \mathfrak{B}(H,N,I \cap H\text{x}N)$ for $q := (H,N) \in Q$, and $\varphi^r_q: L_q \to L_r$, $\psi^r_q: L_r \to L_q$ for $q := (H,N) \leqslant r := (K,P)$ in Q defined by

$\varphi^r_q(A,B) := ((B \cap P)', B \cap P)$ and

$\psi^r_q(C,D) := (C \cap H, (C \cap H)')$.

The table in Fig. 13 describes a context (G,M,I) for the largest linearly representable lattice of factors of an experimental design generated by four factors A,B,C, and D with $C \leqslant D$ (C is nested in D) (21); the relation I given by the crosses is extended to the blockrelation J, indicated by the points in the table. The Q-atlas corresponding to J has the special property that its maps are pairwise disjoint. Therefore the maps of the Q-atlas are represented by disjoint subdiagrams in the line diagram of $\mathfrak{B}(G,M,I)$. In the *divided line diagram* of Fig. 14 (11), for each pair $q < r$ in Q (r covers q in Q), there are two parallel line segments between the subdiagrams representing the maps L_q and L_r; these line segments join 1_q with $\psi^r_q 1_q$ and O_r with $\psi^r_q O_r$, respectively. In order to obtain the complete line diagram from the divided line diagram, we have to draw for all $q < r$ in Q a line segment starting from each x between $\psi^r_q O_r$ and 1_q to $\varphi^r_q x$ parallel to the line segment joining 1_q and $\varphi^r_q 1_q$.

An opposite type of Q-atlas is given by the block relation described in the context of Fig. 15 (17) (21) by the added points; for all $q < r$ in Q we have $L_q \cap L_r \neq \phi$. The maps of the Q-atlas are still represented by disjoint subdiagrams in the divided line diagram in Fig. 16 (21), but the subdiagrams representing L_q and L_r with $q < r$ are now connected by dotted line segments joining the two circles representing 1_q and the two circles representing O_r, respectively. In order to obtain the complete diagram (Fig. 17), we have to push together the subdiagrams along the dotted line segments until the end points of these segments coincide.

In general, a divided line diagram of a sum of a Q-atlas contains straight line segments as well as dotted line segments between its delimited panes, it depends on whether the represented maps have an empty or a non-empty intersection. Further modifications of the line diagram are needed if other lattice constructions are considered; in particular, the tensor product seems noteworthy (22).

4. Conceptual Measurement Theory

In order to clarify the aspect of measurement of the representation by line diagrams, the beginnings of a conceptual measurement theory shall be sketched. In the theory of measurement a scale (a measure) is usually defined as a relatively closed homomorphisme from an empirical relational structure into a numerical relational structure of the same type (13). The approach through

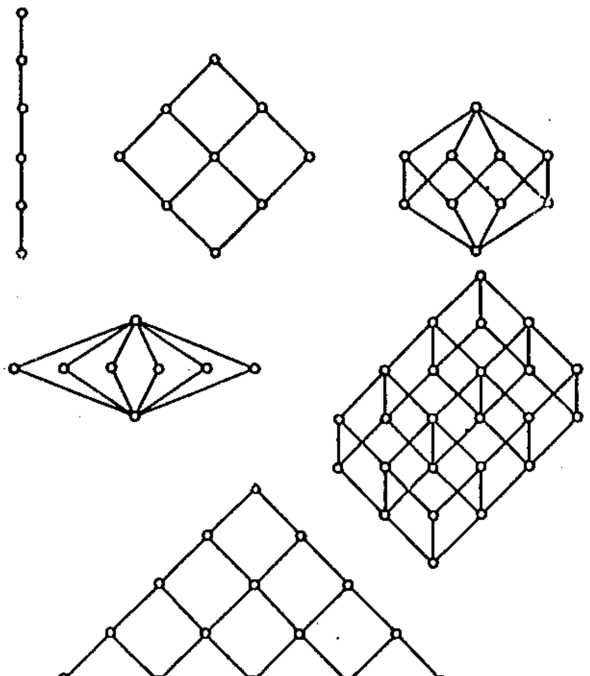

Fig. 18: Examples of graphic scales

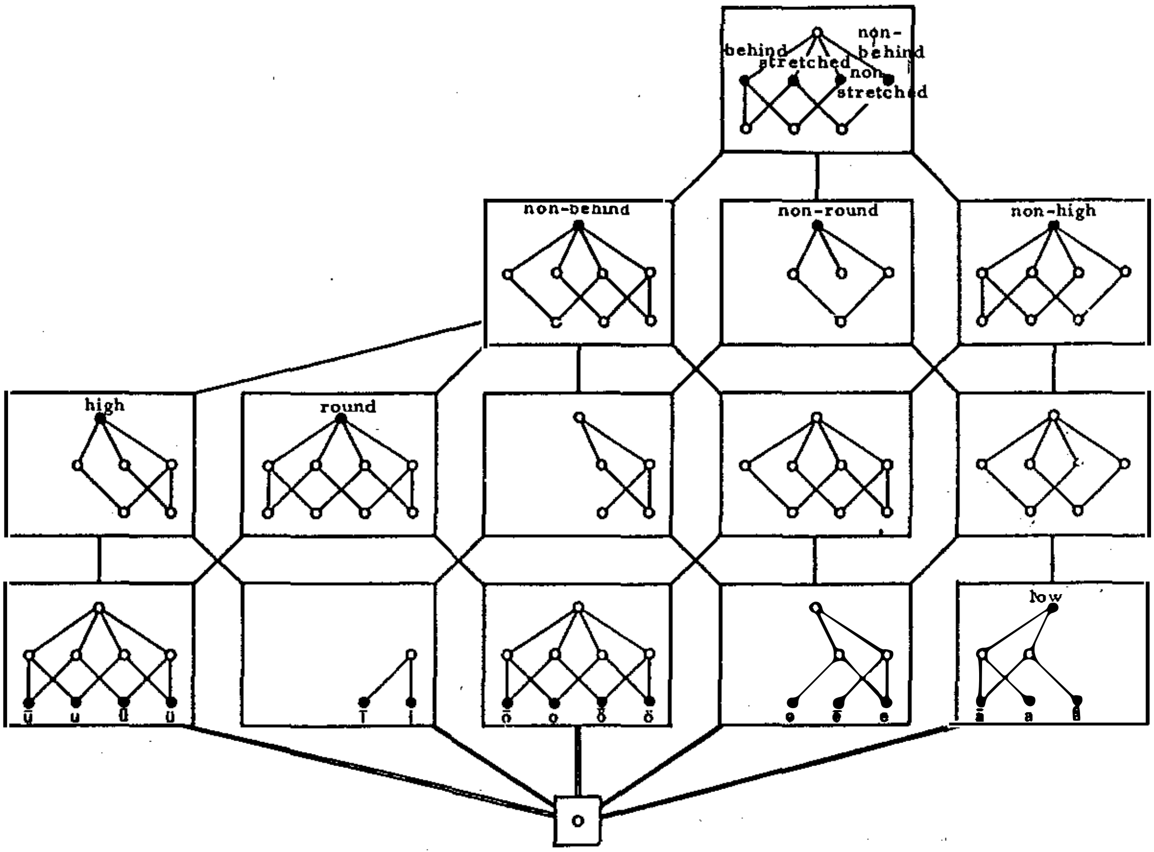

Fig. 19: Context of vowels in the German language together with its concept lattice

	low	non-low	high	non-high	behind	non-behind	round	non-round	stretched	non-stretched
a	x			x	x			x		x
ā	x			x	x			x	x	
i		x	x			x		x		x
ī		x	x			x		x	x	
u		x	x		x		x			x
ū		x	x		x		x		x	
ǟ	x			x		x		x	x	
e		x		x		x		x		x
ē		x		x		x		x	x	
o		x		x	x		x			x
ō		x		x	x		x		x	
ü		x	x			x	x			x
ǖ		x	x			x	x		x	
ö		x		x		x	x			x
ȫ		x		x		x	x		x	
ə		x		x	x			x		x

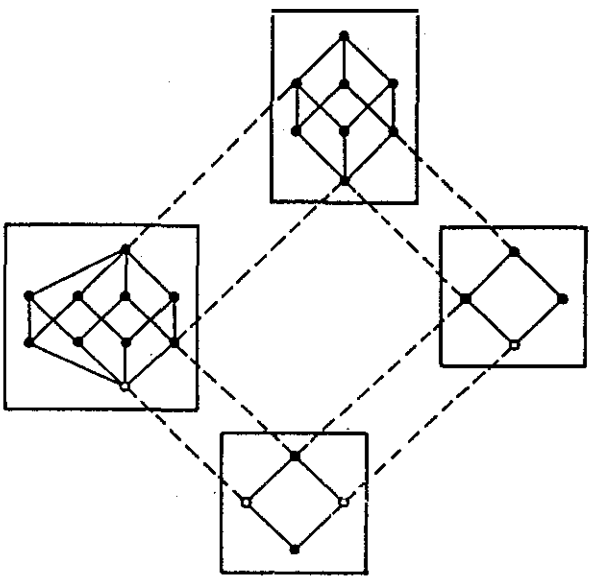

Fig. 20: Graphic scale which underlies the diagram of panes in Fig. 19

concept analysis replaces the typefixed notion of a homomorphism by the type-free notion of a measure from an (empirical) context into a scale (18), in the course of which a *scale* is defined as a context $\$:= (G_\$, M_\$, I_\$)$ for which the concept lattice is "well" known. An $-*measure* of a context (G, M, I) is a mapping σ: $G \to G_\$$ such that, for every extent A of $, the pre-image $\sigma^{-1}A$ is an extent of (G, M, I); σ is said to be full if σ^{-1} includes an isomorphism from $\underline{\mathfrak{B}}(\sigma G, M_\$, I_\$ \cap \sigma G \times M_\$)$ onto $\underline{\mathfrak{B}}(G, M, S)$. Full $-measures can be lattice-theoretically characterized as follows:

Theorem (18) (22): Let $\$:= (G_\$, M_\$, I_\$)$ be a context (a scale) for which $\{g\}' = \{h\}'$ implies $g=h$ $(g,h \in G_\$)$. If σ is a full $-measure of a context (G, M, I) than a $\bigvee$-embedding of $\underline{\mathfrak{B}}(G, M, I)$ into $\underline{\mathfrak{B}}(\$)$ is defined by $\bar{\sigma}(A, B) := ((\sigma A)'', (\sigma A)')$. The mapping $\sigma \mapsto \bar{\sigma}$ is a bijection from the set of all $-measures of (G, M, I) onto the set of all $\bigvee$-embeddings ι from $\underline{\mathfrak{B}}(G, M, I)$ into $\underline{\mathfrak{B}}(\$)$ with the property that for each $g \in G$ there exists an $h \in G_\$$ with $\iota(\{g\}'', \{g\}') = (\{h\}'', \{h\}')$ ($\iota O \neq O$ is admitted).

If L is a complete lattic with a known line diagram and if $\$:= (L, L, \leqslant)$, then each full $-measure σ of a context (G, M, I) yields, since $L \cong \underline{\mathfrak{B}}(\$)$, a $\bigvee$-embedding $\hat{\sigma}$ from $\underline{\mathfrak{B}}(G, M, I)$ into L and thereby a line diagram of $\underline{\mathfrak{B}}(G, M, I)$ which is a subdiagram of the known line diagram of L. L. Thus we may, for instance, consider the line diagrams in the panes of Fig. 8 as the results of $(L,L,\leqslant)$-measures where L is the lattice of all subsets of a four-element set. It seems natural to use the name *graphic scale* for a clear line diagram of a complete lattice L, and to call the representation by a subdiagram corresponding to a $\bigvee$-embedding into L a *measurement* by this graphic scale. Typical examples of graphic scales are shown in Fig. 18.

The problem of drawing larger graphic scales can be solved in many cases by modified line diagrams. A graded line diagram of a concept lattice $\underline{\mathfrak{B}}(G,M,I)$ which is based on a partition $\{M_1, \dots, M_n\}$ of M is established by measurement within the graphic scale which is a graded line diagram of the direct product of the concept lattices $\underline{\mathfrak{B}}(G, M_i, I \cap G \times M_i)$ $(i = 1, \dots, n)$. Therefore a basis of graphic scales is given by the direct products of complete lattices having clear line diagrams (cf. Fig. 18). The sums of atlases of complete lattices may also lead to graphic scales, a fact exemplified by Figs. 19 and 20 (3).

Notes:

1 We mention as sources only (2) and (7) where further references can be found.

2 The "Forschungsgruppe Begriffsanalyse" at the TH Darmstadt is developing a program package for concept analysis for which different algorithms for determining concept lattices are implemented (8).

3 Double line segments join the panes of a graded line diagram which represents a lexicographic decomposition of an ordered set (23).

4 The context in Fig.7 is part of research, the results of which are documented in (14).

5 Answers to the question of how to find suitable block relations are contained in (19), (20), and (21).

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Address: Prof. Dr. Rudolf Wille, Technische Hochschule, Fachbereich Mathematik, Schloßgartenstr. 7, D-6100 Darmstadt